# Econometrics II

# Ordered & Multinomial Outcomes. Tobit regression.

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2 May 2011

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## 1. Introduction

In this lecture we consider the following econometric models:

- Ordered response models (e.g. modelling the rating of the corporate payment default risk, which varies from, say, A (best) to D (worst))
- Multinomial response models (e.g. whether an individual is unemployed, wage-employed or self-employed)
- Corner solution models and censored regression models (e.g. modelling household health expenditure: the dependent variable is non-negative, continuous above zero and has a lot of observations at zero)

These models are designed for situations in which the dependent variable is not strictly continuous and not binary. They can be viewed as extensions of the nonlinear binary choice models studied in Lecture 2 (probit & logit).

References:

Greene 23.10 (ordered response); 23.11 (multinomial response); 24.2-3 (truncation & censored data).

You might also find the following sections in Wooldridge (2002) "Cross Section and Panel Data" useful: 15.9-15.10; 16.1-5; 16.6.3-4; 16.7; 17.3 (these references in Wooldridge are optional).

## 2. Ordered Response Models

What's the meaning of **ordered response**? Consider credit rating on a scale from zero to six, for instance, and suppose this is the variable that we want to model (i.e. this is the dependent variable). Clearly, this is a variable that has ordinal meaning: six is better than five, which is better than four etc.

The standard way of modelling ordered response variables is by means of **ordered probit** or **ordered logit**. These two models are very similar. I will discuss the ordered probit, but everything below carries over to the logit if we replace the normal CDF $\Phi(.)$ by the logistic CDF $\Lambda(.)$.

- Can you think of reasons why OLS may not be suitable for modelling an ordered response variable?

- Could binary choice models (LPM, probit, logit) potentially be used?

### 2.1. Ordered Probit

Let $y$ be an ordered response taking on the values $\{0, 1, 2, ..., J\}$. We derive the ordered probit from a **latent variable model** (cf. probit binary choice)

$$
\begin{aligned}
y^* &= \beta_1 x_1 + ... + \beta_k x_k + \varepsilon \\
&= \boldsymbol{x}'\boldsymbol{\beta} + \varepsilon, \qquad (2.1)
\end{aligned}
$$

where $e$ is a normally distributed variable with the variance normalized to one. Notice that this model does **not** contain a constant. Why will become clear in a moment.

Next define $J$ **cut-off points** (or **threshold parameters**) as follows:

$$\alpha_1 < \alpha_2 < ...\alpha_J.$$

We do not observe the latent variable, but we do observe choices according to the following:

$$
\begin{aligned}
y &= 0 \text{ if } y^* \leq \alpha_1 \\
y &= 1 \text{ if } \alpha_1 < y^* \leq \alpha_2 \\
y &= 2 \text{ if } \alpha_2 < y^* \leq \alpha_3 \\
& (...) \\
y &= J \text{ if } \alpha_J < y^*.
\end{aligned}
$$

Think of the cut-off points as intercept shifters. This is how Stata specifies the model. Greene does it slightly differently, including an intercept in the $\boldsymbol{x}$ vector and (effectively) normalizes $\alpha_1 = 0$.

Suppose $y$ can take three values: 0, 1 or 2. We then have

$$\begin{aligned}
y &= 0 \text{ if } \boldsymbol{x}'\boldsymbol{\beta} + \varepsilon \leq \alpha_1 \\
y &= 1 \text{ if } \alpha_1 < \boldsymbol{x}'\boldsymbol{\beta} + \varepsilon \leq \alpha_2 \\
y &= 2 \text{ if } \alpha_2 < \boldsymbol{x}'\boldsymbol{\beta} + \varepsilon.
\end{aligned}$$

We can now define the probabilities of observing $y = 0, 1, 2$. For the smallest and the largest value, the resulting expressions are very similar to what we have seen for the binary probit:

$$\begin{aligned}
\Pr(y = 0|x) &= \Pr(\boldsymbol{x}'\boldsymbol{\beta} + \varepsilon \leq \alpha_1) \\
&= \Pr(e \leq \alpha_1 - \boldsymbol{x}'\boldsymbol{\beta}) \\
&= \Phi(\alpha_1 - \boldsymbol{x}'\boldsymbol{\beta}), \\
&= 1 - \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_1) \\
\Pr(y = 2|x) &= \Pr(\boldsymbol{x}'\boldsymbol{\beta} + \varepsilon > \alpha_2) \\
&= \Pr(\varepsilon > \alpha_2 - \boldsymbol{x}'\boldsymbol{\beta}) \\
&= 1 - \Phi(\alpha_2 - \boldsymbol{x}'\boldsymbol{\beta}) \\
&= \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_2).
\end{aligned}$$

For the intermediate category, we get:

$$\begin{aligned}
\Pr(y = 1|x) &= \Pr(\alpha_1 < \boldsymbol{x}'\boldsymbol{\beta} + \varepsilon \leq \alpha_2) \\
&= \Pr(\varepsilon > \alpha_1 - \boldsymbol{x}'\boldsymbol{\beta}, \varepsilon \leq \alpha_2 - \boldsymbol{x}'\boldsymbol{\beta}) \\
&= [1 - \Phi(\alpha_1 - \boldsymbol{x}'\boldsymbol{\beta})] - \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_2), \\
&= 1 - (1 - \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_1)) - \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_2), \\
&= \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_1) - \Phi(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_2),
\end{aligned}$$

or equivalently

$$\Pr\left(y=1|x\right)=\Phi\left(\alpha_2-\boldsymbol{x}'\boldsymbol{\beta}\right)-\Phi\left(\alpha_1-\boldsymbol{x}'\boldsymbol{\beta}\right)$$

(remember: $\Phi(a)=1-\Phi(-a)$, because the normal distribution is symmetric - keep this in mind when studying ordered probits or you might get lost in the algebra). In the general case where there are several intermediate categories, all the associated probabilities will be of this form; see Greene, p.832. Notice that the probabilities sum to one.

### 2.2. Interpretation

When discussing binary choice models we paid a lot of attention to marginal effects - i.e. the partial effects of a small change in explanatory variable $x_j$ on the probability that we have a positive outcome.

For ordered models, we can clearly compute marginal effects on the predicted probabilities along the same principles. It is not obvious (to me, anyway) that this the most useful way of interpreting the results is, however. Let's have a look the marginal effects and then discuss.

#### 2.2.1. Partial effects on predicted probabilities

When discussing marginal effects for binary choice models, we focussed on the effects on the probability that $y$ (the binary dependent variable) is equal to one. We ignored discussing effects on the probability that $y$ is equal to zero, as these will always be equal to minus one times the partial effect on the probability that $y$ is equal to one.

Since we now have more than two outcomes, interpretation of partial effects on probabilities becomes somewhat more awkward. Sticking to the example in which we have three possible outcomes, we obtain:

$$\frac{\partial \Pr\left(y=2|x\right)}{\partial x_k}=\phi\left(\boldsymbol{x}'\boldsymbol{\beta}-\alpha_2\right)\beta_k,$$

for the highest category (note: analogous to the expression for binary probit).[1] Moreover,

$$\frac{\partial \Pr(y=1|x)}{\partial x_k} = \left[\phi\left(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_1\right) - \phi\left(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_2\right)\right]\beta_k,$$

for the intermediate category, and

$$\frac{\partial \Pr(y=0|x)}{\partial x_k} = -\phi\left(\boldsymbol{x}'\boldsymbol{\beta} - \alpha_1\right)\beta_k,$$

for the lowest category, assuming that $x_k$ is a continuous variable enter the index model linearly (if $x_k$ is discrete - typically binary - you just compute the discrete change in the predicted probabilities associated with changing $x_k$ by one unit, for example from 0 to 1). We observe:

- The partial effect of $x_k$ on the predicted probability of the highest outcome has the same sign as $\beta_k$.
- The partial effect of $x_k$ on the predicted probability of the lowest outcome has the opposite sign to $\beta_k$
- The sign of the partial effect of $x_k$ on predicted probabilities of intermediate outcomes cannot, in general, be inferred from the sign of $\beta_k$. This is because there are two offsetting effects - suppose $\beta_k > 0$, then the intermediate category becomes more likely if you increase $x_k$ because the the probability of the lowest category falls, but it also becomes less likely because the the probability of the highest category increases (illustrate this in a graph). Typically, partial effects for intermediate probabilities are quantitatively small and often statistically insignificant. Don't let this confuse you!

**Discussion - how best interpret results from ordered probit (or logit)?**

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[1] Remember that $\phi(a) = \phi(-a)$ - i.e. I could just as well have written

$$\frac{\partial \Pr(y=2|x)}{\partial x_k} = \phi(\alpha_2 - \beta x)\beta_k,$$

for instance - cf. Greene's exposition on p.833.

- Clearly one option here is to look at the estimated $\beta$-parameters, emphasizing the underlying **latent variable equation** with which we started. Note that we don't identify the standard deviation of $\varepsilon$ separately. Note also that consistent estimation of the $\beta$-parameters requires the model to be correctly specified - e.g. homoskedasticity and normality need to hold, if we are using ordered probit. Such assumptions are testable using, for example, the methods introduced for binary choice models. You don't often see this done in applied work however.

- Another option might be to look at the effect on the **expected value** of the ordered response variable, e.g.

$$\frac{\partial E(y|x,\beta)}{\partial x_k} = \frac{\partial \Pr(y=0|x)}{\partial x_k} \times 0 + \frac{\partial \Pr(y=1|x)}{\partial x_k} \times 1 + \frac{\partial \Pr(y=2|x)}{\partial x_k} \times 2,$$

  in our example with three possible outcomes. This may make a lot of sense if $y$ is a numerical variable - basically, if you are prepared to compute mean values of $y$ and interpret them. For example, suppose you've done a survey measuring consumer satisfaction where 1="very unhappy", 2="somewhat unhappy", 3="neither happy nor unhappy", 4="somewhat happy", and 5="very happy", then most people would be prepared to look a the sample mean even though strictly the underlying variable is qualitative, thinking that 3.5 (for example) means something (consumers are on average a little bit happy?). In such a case you could look at partial effects on the conditional mean.

- Alternatively, you might want investigate the effect on the probability of observing categories $j, j+1, ..., J$. In my consumer satisfaction example, it would be straightforward to compute the partial effect on the probability that a consumer is "somewhat happy" or "very happy", for example.

- Thus, it all boils down to presentation and interpretation here, and exactly what your quantity of interest is depends on the context. We can use the Stata command 'mfx compute' to obtain estimates of the partial effects on the predicted probabilities, but for more elaborate partial effects you may have to do some coding tailored to the context.

EXAMPLE: Incidence of corruption in Kenyan firms. Section 1 in the appendix.

## 3. Multinomial response: Multinomial logit

Suppose now the dependent variable is such that more than two outcomes are possible, where the outcomes cannot be ordered in any natural way. For example, suppose we are modelling occupational status based on household data, where the possible outcomes are self-employed (SE), wage-employed (WE) or unemployed (UE). Alternatively, suppose we are modelling the transportation mode for commuting to work: bus, train, car,...

Binary probit and logit models are ill suited for modelling data of this kind. Of course, in principle we could bunch two or more categories and so construct a binary outcome variable from the raw data (e.g. if we don't care if employed individuals are self-employed or wage-employees, we may decide to construct a binary variable indicating whether someone is unemployed or employed). But in doing so, we throw away potentially interesting information. And OLS is obviously not a good model in this context.

However, the logit model for binary choice can be **extended** to model more than two outcomes. Suppose there are $J$ possible outcomes in the data. The dependent variable $y$ can then take $J$ values, e.g. 0,1,...,J-1. So if we are modelling, say, occupational status, and this is either SE, WE or UE, we have $J = 3$. There is no natural ordering of these outcomes, and so what number goes with what category is arbitrary (but, as we shall see, it matters for the interpretation of the results). Suppose we decide on the following:

$$\begin{aligned} y &= 0 \text{ if individual is UE,} \\ y &= 1 \text{ if individual is WE,} \\ y &= 2 \text{ if individual is SE.} \end{aligned}$$

We write the conditional probability that an individual belongs to category $j = 0, 1, 2$ as

$$\Pr(y_i = j | \boldsymbol{x}_i),$$

where $\boldsymbol{x}_i$ is a vector of explanatory variables. Reasonable restrictions on these probabilities are:

- that each of them is bounded in the (0,1) interval,
- that they sum to unity (one).

One way of imposing these restrictions is to write the probabilities in logit form:

$$\Pr(y_i = 1 | \boldsymbol{x}_i) = \frac{\exp(\boldsymbol{x}_i \boldsymbol{\beta}_1)}{1 + \exp(\boldsymbol{x}_i \boldsymbol{\beta}_1) + \exp(\boldsymbol{x}_i \boldsymbol{\beta}_2)},$$

$$\Pr(y_i = 2 | \boldsymbol{x}_i) = \frac{\exp(\boldsymbol{x}_i \boldsymbol{\beta}_2)}{1 + \exp(\boldsymbol{x}_i \boldsymbol{\beta}_1) + \exp(\boldsymbol{x}_i \boldsymbol{\beta}_2)},$$

$$\begin{aligned} \Pr(y_i = 0 | \boldsymbol{x}_i) &= 1 - \Pr(y_i = 1 | \boldsymbol{x}_i) - \Pr(y_i = 2 | \boldsymbol{x}_i) \\ &= \frac{1}{1 + \exp(\boldsymbol{x}_i \boldsymbol{\beta}_1) + \exp(\boldsymbol{x}_i \boldsymbol{\beta}_2)}. \end{aligned}$$

The main difference compared to what we have seen so far, is that there are now **two** parameter vectors, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ (in the general case with $J$ possible responses, there are $J - 1$ parameter vectors). This makes interpretation of the coefficients more difficult than for binary choice models.

- The easiest case to think about is where $\beta_{1k}$ and $\beta_{2k}$ have the same sign. If $\beta_{1k}$ and $\beta_{2k}$ are positive (negative) then it is clear that an increase in the variable $x_k$ makes it less (more) likely that the individual belongs to category 0.
- But what about the effects on $\Pr(y_i = 1 | \boldsymbol{x}_i)$ and $\Pr(y_i = 2 | \boldsymbol{x}_i)$? This is much trickier than what we are used to. We know that, for sure, the sum of $\Pr(y_i = 1 | \boldsymbol{x}_i)$ and $\Pr(y_i = 2 | \boldsymbol{x}_i)$ will increase, but how this total increase is allocated between these two probabilities is not obvious. To find out,

we need to look at the marginal effects. We have

$$\begin{aligned}\frac{\partial \Pr(y_i=1|\boldsymbol{x}_i)}{\partial x_{ik}} &= \beta_{1k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)\left[1+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)\right]^{-1} \\ &\quad -\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)\left[1+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)\right]^{-2} \\ &\quad \times\left(\beta_{1k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\beta_{2k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)\right),\end{aligned}$$

which can be written as

$$\begin{aligned}\frac{\partial \Pr(y_i=1|\boldsymbol{x}_i)}{\partial x_{ik}} &= \beta_{1k}\Pr(y_i=1|\boldsymbol{x}_i) \\ &\quad -\Pr(y_i=1|\boldsymbol{x}_i)\left[1+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)\right]^{-1} \\ &\quad \times\left(\beta_{1k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\beta_{2k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)\right),\end{aligned}$$

or

$$\frac{\partial \Pr(y_i=1|\boldsymbol{x}_i)}{\partial x_{ik}} = \Pr(y_i=1|\boldsymbol{x}_i)\left[\beta_{1k}-\frac{\beta_{1k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\beta_{2k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)}{1+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)}\right]. \tag{3.1}$$

Similarly, for $j=2$:

$$\frac{\partial \Pr(y_i=2|\boldsymbol{x}_i)}{\partial x_{ik}} = \Pr(y_i=2|\boldsymbol{x}_i)\left[\beta_{2k}-\frac{\beta_{1k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\beta_{2k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)}{1+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)}\right],$$

while for the base category $j=0$:

$$\frac{\partial \Pr(y_i=0|\boldsymbol{x}_i)}{\partial x_{ik}} = \Pr(y_i=0|\boldsymbol{x}_i)\left[-\frac{\beta_{1k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\beta_{2k}\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)}{1+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)+\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)}\right].$$

Of course it's virtually impossible to remember, or indeed interpret, these expressions. The point is that whether the probability that $y$ falls into, say, category 1 rises or falls as a result of varying $x_{ik}$, depends not only on the parameter estimate $\beta_{1k}$, but also on $\beta_{2k}$. As you can see from (3.1), the marginal effect $\frac{\partial \Pr(y_i=1|\boldsymbol{x}_i)}{\partial x_{ik}}$ may in fact be negative even if $\beta_{1k}$ is positive, and vice versa. Why might that happen?

- EXAMPLE: Appendix, Section 2. Occupational outcomes amongst Kenyan manufacturing workers.

#### 3.0.2. Independence of irrelevant alternatives (IIA)

Reference: Greene, 23.11.3.

The multinomial logit is very convenient for modelling an unordered discrete variable that can take on more than two values. One important limitation of the model is that the ratio of any two probabilities $j$ and $m$ depends **only** on the parameter vectors $\beta_j$ and $\beta_m$, and the explanatory variables $x_i$:

$$\begin{aligned}\frac{\Pr(y_i = 1|\boldsymbol{x}_i)}{\Pr(y_i = 2|\boldsymbol{x}_i)} &= \frac{\exp(\boldsymbol{x}_i\boldsymbol{\beta}_1)}{\exp(\boldsymbol{x}_i\boldsymbol{\beta}_2)} \\ &= \exp(\boldsymbol{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)).\end{aligned}$$

It follows that the inclusion or exclusion of **other categories** must be irrelevant to the ratio of the two probabilities that $y = 1$ and $y = 2$. This is potentially restrictive, in a behavioral sense.

**Example:** Individuals can commute to work by three transportation means: blue bus, red bus, or train. Individuals choose one of these alternatives, and the econometrician estimates a multinomial logit modelling this decision, and obtains an estimate of

$$\frac{\Pr(y_i = \text{red bus}|\boldsymbol{x}_i)}{\Pr(y_i = \text{train}|\boldsymbol{x}_i)}.$$

Suppose the bus company were to remove blue bus from the set of options, so that individuals can choose only between red bus and train. If the econometrician were to estimate the multinomial logit on data generated under this regime, do you think the above probability ratio would be the same as before?

If not, this suggests the multinomial logit modelling the choice between blue bus, red bus and train is mis-specified: the presence of a blue bus alternative is not irrelevant for the above probability ratio, and thus for individuals' decisions more generally.

Some authors (e.g. Greene; Stata manuals) claim we can **test** the IIA assumption for the multinomial logit by means of a Hausman test. The basic idea is as follows:

1. Estimate the full model. For example, with red bus, train and blue bus being the possible outcomes, and with red bus defined as the benchmark category. Retain the coefficient estimates.

2. Omit one category and re-estimate the model - e.g. exclude blue bus, and model the binary decision to go by train as distinct from red bus.

3. Compare the coefficients from (1) and (2) above using the usual Hausman formula. Under the null that IIA holds, the coefficients should not be significantly different from each other.

This procedure does not make sense to me.

- First, you don't really have data generated in the alternative regime (with blue bus not being an option) and so how can you hope to shed light on the behavioral effect of removing blue bus from the set of options?

- Second, obviously **sample means** of ratios such as

$$\frac{\Pr(y_i = \text{red bus})}{\Pr(y_i = \text{train})} = \frac{N_{\text{red bus}}/N}{N_{\text{train}}/N}$$

  don't depend on blue bus outcomes. So if you estimate a multinomial logit with only a constant included in the specification, the estimated constant in the specification train specification (with red bus as the reference outcome) will not change if you omit blue bus outcomes when estimating (i.e. step (2) above). Conceptually, a similar issue will hold if you have explanatory variables in the model, at least if you have a flexible functional form in your $\boldsymbol{x}_i\boldsymbol{\beta}$ indices (e.g. mutually exclusive dummy variables)

- Third, from what I have seen the Hausman test for the IIA does not work well in practice (not very surprising).

- While testing for IIA in the context of a multinomial logit appears problematic, it may more sense in a different setting. For example, it will work fine for **conditional** logit models, i.e. models where choices are made based on observable attributes of each alternative (e.g. ticket prices for blue bus, red bus and train may vary). So, what I have said above applies specifically for the multinomial logit.

Note that there are lots of other econometric models that can be used to model multinomial response models - notably multinomial probit, conditional logit, nested logit etc. These will not be discussed here.

EXAMPLE: Hausman test for IIA based on multinomial logit gives you nonsense - appendix, Section 3.

## 4. Tobit Estimation of Corner Solution Models

Reference: Greene. 24.2-3.

We now consider econometric issues that arise when the dependent variable is bounded but continuous within the bounds. We focus first on corner solution models, and then turn to the censored regression model (duration data is often censored) and truncated regression.

In general, a **corner solution response variable** is bounded such that

$$lo \leq y_i \leq hi,$$

where $lo$ denotes the lower bound (limit) and $hi$ the higher bound, and where these bounds are the result of real economic constraints.

- By far the most common case is $lo = 0$ and $hi = \infty$, i.e. there is a lower limit at zero and no upper limit. The dependent variable takes the value zero for a nontrivial fraction of the population, and is roughly continuously distributed over positive values. You will often find this in micro data, e.g. household expenditure on education, health, alcohol,... .

- You can thus think of this type of variable as a **hybrid** between a continuous variable (for which the linear model is appropriate) and a binary variable (for which one would typically use a binary choice model). Indeed, as we shall see, the econometric model designed to model corner solution variables looks like a hybrid between OLS and the probit model. In what follows we focus on the case where $lo = 0$, $hi = \infty$, however generalizing beyond this case is reasonably straightforward.

Let $y$ be a variable that is equal to zero for some non-zero proportion of the population, and that is continuous and positive if it is not equal to zero. As usual, we want to model $y$ as a function of a set of variables $x_1, x_2, ..., x_k$ - or in matrix notation:

$$\boldsymbol{x} = \begin{bmatrix} 1 & x_1 & x_2 & ... & x_k \end{bmatrix}.$$

### 4.1. OLS

We have seen how for binary choice models OLS can be a useful starting point (yielding the linear probability model), even though the dependent variable is not continuous. We now have a variable which is 'closer' to being a continuous variable - it's discrete in the sense that it is either in the corner (equal to zero) or not (in which case it's continuous).

OLS is a useful starting point for modelling corner solution variables:

$$y = \boldsymbol{x\beta} + u.$$

We've seen that there are a number of reasons why we may not prefer to estimate binary choice models using OLS. For similar reasons OLS may not be an ideal estimator for corner response models:

- Based on OLS estimates we can get **negative predictions**, which doesn't make sense since the dependent variable is non-negative (if we are modelling household expenditure on education, for instance, negative predicted values do not make sense).

- Conceptually, the idea that a corner solution variable is **linearly** related to a continuous independent variable for all possible values is a bit suspect. It seems more likely that for observations close to the corner (close to zero), changes in some continuous explanatory variable (say $x_1$) has a smaller effect on the outcome than for observations far away from the corner. So if we are interested in understanding how $y$ depends on $x_1$ among low values of $y$, linearity is not attractive.

- A third (and less serious) problem is that the residual $u$ is likely to be heteroskedastic - but we can deal with this by simply correcting the standard errors.

- A fourth and related problem is that, because the distribution of $y$ has a 'spike' at zero, the residual cannot be normally distributed. This means that OLS point estimates are unbiased, but inference in small samples cannot be based on the usual suite of normality-based distributions such as the $t$ test.

So you see all of this is very similar to the problems identified with the linear probability model.

### 4.2. Tobit

To fix these problems we follow a similar path as for binary choice models. We start, however, from the latent variable model, written as

$$y^* = \boldsymbol{x\beta} + u, \tag{4.1}$$

where the residual $u$ is assumed **normally distributed** with a **constant variance** $\sigma_u^2$, and uncorrelated with $\boldsymbol{x}$**.** As usual, the latent variable $y^*$ is unobserved - we observe

$$y = \left\{ \begin{array}{c} y^* \text{ if } y^* > 0 \\ 0 \text{ if } y^* \leq 0 \end{array} \right\}, \tag{4.2}$$

which can be written equivalently as

$$y = \max(y^*, 0).$$

Two things should be noted here.

- First, $y^*$ satisfies the classical linear model assumptions, so had $y^*$ been observed the obvious choice of estimator would have been OLS.

- Second, it is often helpful to think of $y$ as a variable that is bounded below for **economic** reasons, and $y^*$ as a variable that reflects the 'desired' value if there were no constraints. Actual household expenditure on health is one example - this is bounded below at zero. In such a case $y^*$ could be interpreted as desired expenditure, in which case $y^* < 0$ would reflect a desire to sell off ones personal (or family's) health. This may not be as far-fetched as it sounds - if you're very healthy and very poor, for instance, perhaps you wouldn't mind feeling a little less healthy if you got paid for it (getting paid here, of course, would be the same as having negative health expenditure).

We said above that a corner solution variable is a kind of hybrid: both discrete and continuous. The discrete part is due to the piling up of observations at zero. The probability that $y$ is equal to zero can be written

$$\begin{aligned}
\Pr(y=0|x) &= \Pr(y^* \leq 0), \\
&= \Pr(\boldsymbol{x\beta} + u \leq 0), \\
&= \Pr(u \leq -\boldsymbol{x\beta}) \\
&= \Phi\left(\frac{-\boldsymbol{x\beta}}{\sigma_u}\right) \text{ (integrate; normal distribution)} \\
\Pr(y=0|x) &= 1-\Phi\left(\frac{\boldsymbol{x\beta}}{\sigma_u}\right) \text{ (by symmetry),}
\end{aligned}$$

exactly like the probit model. In contrast, if $y > 0$ then it is continuous:

$$y = \boldsymbol{x\beta} + u.$$

It follows that the conditional density of $y$ is equal to

$$f(y|\boldsymbol{x};\boldsymbol{\beta},\sigma_u) = [1-\Phi(\boldsymbol{x}_i\boldsymbol{\beta}/\sigma_u)]^{1_{[y(i)=0]}} \left[\phi\left(\frac{y_i - \boldsymbol{x}_i\boldsymbol{\beta}}{\sigma_u}\right)\right]^{1_{[y(i)>0]}},$$

where $1_{[a]}$ is a dummy variable equal to one if $a$ is true. Thus the contribution of observation $i$ to the sample log likelihood is

$$\ln L_i = 1_{[y(i)=0]} \ln \left[1 - \Phi\left(\boldsymbol{x}_i \boldsymbol{\beta}/\sigma_u\right)\right] + 1_{[y(i)>0]} \ln \left[\phi\left(\frac{y_i - \boldsymbol{x}_i \boldsymbol{\beta}}{\sigma_u}\right)\right],$$

and the sample log likelihood is

$$\ln L\left(\boldsymbol{\beta}, \sigma_u\right) = \sum_{i=1}^{N} \ln L_i.$$

Estimation is done by means of maximum likelihood.

### 4.2.1. Interpreting the tobit model

Suppose the model can be written according to the equations (4.1)-(4.2), and suppose we have obtained estimates of the parameter vector $\boldsymbol{\beta}$. How do we interpret these parameters?

We see straight away from the latent variable model that $\beta_j$ is interpretable as the partial (marginal) effects of $x_j$ on the latent variable $y^*$, i.e.

$$\frac{\partial E\left(y^* | \boldsymbol{x}\right)}{\partial x_j} = \beta_j,$$

if $x_j$ is a continuous variable, and

$$E\left(y^* | x_j = 1\right) - E\left(y^* | x_j = 0\right) = \beta_j$$

if $x_j$ is a dummy variable (of course if $x_j$ enters the model nonlinearly these expressions need to be modified accordingly). I have omitted $i$-subscripts for simplicity. If that's what we want to know, then we are home: all we need is an estimate of the relevant parameter $\beta_j$.

Typically, however, we are interested in the partial effect of $x_j$ on the expected **actual outcome** $y$, rather than on the latent variable. Think about the health example above. We are probably primarily interested in the partial effects of $x_j$ (perhaps household size) on expected actual - rather than desired

- health expenditure, e.g. $\partial E(y|\boldsymbol{x})/\partial x_j$ if $x_j$ is continuous. In fact there are two different potentially interesting marginal effects, namely

$$\frac{\partial E(y|\boldsymbol{x})}{\partial x_j}, \qquad \text{(Unconditional on y)}$$

and

$$\frac{\partial E(y|\boldsymbol{x}, y>0)}{\partial x_j}. \qquad \text{(Conditional on y>0)}$$

We need to be clear on which of these we are interested in. Now let's see what these marginal effects look like.

**The marginal effects on expected $y$, conditional on $y$ positive.** We want to derive

$$\frac{\partial E(y|\boldsymbol{x}, y>0)}{\partial x_j}.$$

Recall that the model can be written

$$\begin{aligned} y &= \max(y^*, 0), \\ y &= \max(\boldsymbol{x\beta} + u, 0) \end{aligned}$$

(see (4.1)-(4.2)). We begin by writing down $E(y|x, y>0)$:

$$\begin{aligned} E(y|y>0, \boldsymbol{x}) &= E(\boldsymbol{x\beta} + u|y>0, \boldsymbol{x}), \\ E(y|y>0, \boldsymbol{x}) &= \boldsymbol{x\beta} + E(u|y>0, \boldsymbol{x}), \\ E(y|y>0, \boldsymbol{x}) &= \boldsymbol{x\beta} + E(u|u>-\boldsymbol{x\beta}) \end{aligned}$$

Because of the truncation ($y$ is always positive, or, equivalently, $u$ is always larger than $-\boldsymbol{x\beta}$), dealing with the second term is not as easy as it may seem. We begin by taking on board the following result for normally distributed variables:

- **A useful result.** If $z$ follows a normal distribution with mean zero, and variance equal to one (i.e. a standard normal distribution), then

$$E\left(z|z>c\right)=\frac{\phi\left(c\right)}{1-\Phi\left(c\right)}, \tag{4.3}$$

where $c$ is a constant (i.e. the lower bound here), $\phi$ denotes the standard normal probability density, and $\Phi$ is the standard normal cumulative density.

The residual $u$ is not, in general, standard normal because the variance is not necessarily equal to one, but by judiciously dividing and multiplying through with its standard deviation $\sigma_u$ we can transform $u$ to become standard normal:

$$E\left(y|y>0,x\right)=\boldsymbol{x\beta}+\sigma_u E\left(u/\sigma_u|u/\sigma_u>-\boldsymbol{x\beta}/\sigma_u\right).$$

That is, $(u/\sigma_u)$ is now standard normal, and so we can apply the above 'useful result', i.e. eq (4.3), and write:

$$E\left(u|u>-\boldsymbol{x\beta}\right)=\sigma_u\frac{\phi\left(-\boldsymbol{x\beta}/\sigma_u\right)}{1-\Phi\left(-\boldsymbol{x\beta}/\sigma_u\right)},$$

and thus

$$E\left(y|y>0,\boldsymbol{x}\right)=\boldsymbol{x\beta}+\sigma_u\frac{\phi\left(-\boldsymbol{x\beta}/\sigma_u\right)}{1-\Phi\left(-\boldsymbol{x\beta}/\sigma_u\right)}.$$

With slightly cleaner notation,

$$E\left(y|y>0,\boldsymbol{x}\right)=\boldsymbol{x\beta}+\sigma_u\frac{\phi\left(\boldsymbol{x\beta}/\sigma_u\right)}{\Phi\left(\boldsymbol{x\beta}/\sigma_u\right)},$$

which is often written as

$$E\left(y|y>0,\boldsymbol{x}\right)=\boldsymbol{x\beta}+\sigma_u\lambda\left(\boldsymbol{x\beta}/\sigma_u\right), \tag{4.4}$$

where the function $\lambda$ is defined as

$$\lambda(z) = \frac{\phi(z)}{\Phi(z)}.$$

in general, and known as the **inverse Mills ratio** function.

- Have a look at the inverse Mills ratio function in Section 4 in the appendix, Figure 1.

Equation (4.4) shows that the expected value of $y$, given that $y$ is not zero, is equal to $\boldsymbol{x\beta}$ **plus** a term $\sigma_u \lambda(\boldsymbol{x\beta}/\sigma_u)$ which is strictly positive (how do we know that?).

We can now obtain the marginal effect:

$$\begin{aligned} \frac{\partial E(y|y>0,\boldsymbol{x})}{\partial x_j} &= \beta_j + \sigma_u \frac{\partial \lambda(\boldsymbol{x\beta}/\sigma_u)}{\partial x_j}, \\ &= \beta_j + \sigma_u \left(\beta_j/\sigma_u\right) \lambda', \\ &= \beta_j \left(1+\lambda'\right), \end{aligned}$$

where $\lambda'$ denotes the partial derivative of $\lambda$ with respect to $(\boldsymbol{x\beta}/\sigma_u)$ (note: I am assuming here that $x_j$ is continuous and not functionally related to any other variable - i.e. it enters the model linearly - this means I can use calculus, and that I don't have to worry about higher-order terms). It is tedious but fairly easy to show that

$$\lambda'(z) = -\lambda(z)[z+\lambda(z)]$$

in general, hence

$$\frac{\partial E(y|y>0,\boldsymbol{x})}{\partial x_j} = \beta_j \left\{1 - \lambda(\boldsymbol{x\beta}/\sigma_u)\left[\boldsymbol{x\beta}/\sigma_u + \lambda(\boldsymbol{x\beta}/\sigma_u)\right]\right\}.$$

This shows that the partial effect of $x_j$ on $E(y|y>0,\boldsymbol{x})$ is not determined just by $\beta_j$. In fact, it depends on **all parameters $\boldsymbol{\beta}$** in the model as well as on the values of **all explanatory variables $\boldsymbol{x}$**, and the standard deviation of the residual. The term in $\{\cdot\}$ is often referred to as the **adjustment factor**, and it can be shown that this is always larger than zero and smaller than one (why is this useful to know?).

It should be clear that, just as in the case for probits and logits, we need to evaluate the marginal effects at specific values of the explanatory variables. This should come as no surprise, since one of the reasons we may prefer tobit to OLS is that we have reasons to believe the marginal effects may differ according to how close to the corner (zero) a given observation is (see above). In Stata we can use the *mfx compute* command to compute marginal effects without too much effort. How this is done will be clearer in a moment, but first I want to go over the second type of marginal effect that I might be interested in.

**The marginal effects on expected $y$, unconditional on the value of $y$** Recall:

$$\begin{aligned} y &= \max\left(y^{*}, 0\right), \\ y &= \max\left(\boldsymbol{x}\boldsymbol{\beta} + u, 0\right). \end{aligned}$$

I now need to derive

$$\frac{\partial E\left(y|\boldsymbol{x}\right)}{\partial x_j}.$$

We write $E\left(y|\boldsymbol{x}\right)$ as follows:

$$\begin{aligned} E\left(y|x\right) &= \Phi\left(-\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \cdot E\left(y|y=0,\boldsymbol{x}\right) + \Phi\left(\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \cdot E\left(y|y>0,\boldsymbol{x}\right), \\ &= \Phi\left(-\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \cdot 0 + \Phi\left(\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \cdot E\left(y|y>0,\boldsymbol{x}\right), \\ &= \Phi\left(\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \cdot E\left(y|y>0,\boldsymbol{x}\right), \end{aligned}$$

i.e. the probability that $y$ is positive times the expected value of $y$ given that $y$ is indeed positive. Using the product rule for differentiation,

$$\frac{\partial E\left(y|x\right)}{\partial x_j} = \Phi\left(\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \cdot \frac{\partial E\left(y|y>0,\boldsymbol{x}\right)}{\partial x_j} + \phi\left(\boldsymbol{x}\boldsymbol{\beta}/\sigma_u\right) \frac{\beta_j}{\sigma_u} \cdot E\left(y|y>0,\boldsymbol{x}\right),$$

and we know from the previous sub-section that

$$\frac{\partial E\left(y|y>0,\boldsymbol{x}\right)}{\partial x_j}=\beta_j\left\{1-\lambda\left(\boldsymbol{x\beta}/\sigma_u\right)\left[\boldsymbol{x\beta}/\sigma_u+\lambda\left(\boldsymbol{x\beta}/\sigma_u\right)\right]\right\},$$

and

$$E\left(y|y>0,\boldsymbol{x}\right)=\boldsymbol{x\beta}+\sigma_u\lambda\left(\boldsymbol{x\beta}/\sigma_u\right).$$

Hence

$$\begin{aligned}\frac{\partial E\left(y|\boldsymbol{x}\right)}{\partial x_j} &= \Phi\left(\boldsymbol{x\beta}/\sigma_u\right)\cdot\beta_j\left\{1-\lambda\left(\boldsymbol{x\beta}/\sigma_u\right)\left[\boldsymbol{x\beta}/\sigma_u+\lambda\left(\boldsymbol{x\beta}/\sigma_u\right)\right]\right\}\\ &\quad+\phi\left(\boldsymbol{x\beta}/\sigma_u\right)\frac{\beta_j}{\sigma_u}\cdot\left[\boldsymbol{x\beta}+\sigma_u\lambda\left(\boldsymbol{x\beta}/\sigma_u\right)\right],\end{aligned}$$

which looks complicated but the good news is that several of the terms cancel out, so that:

$$\frac{\partial E\left(y|\boldsymbol{x}\right)}{\partial x_j}=\beta_j\Phi\left(\boldsymbol{x\beta}/\sigma_u\right)$$

(try to prove this). This has a straightforward interpretation: the marginal effect of $x_j$ on the expected value of $y$, conditional on the vector $\boldsymbol{x}$, is simply the parameter $\beta_j$ times the probability that $y$ is larger than zero. Of course, this probability is smaller than one, so it follows immediately that the marginal effect is strictly smaller than the parameter $\beta_j$.

Now consider the example in section 4 in the appendix, on investment in plant and machinery among Ghanaian manufacturing firms.

**PhD Programme: Econometrics II**
**Department of Economics, University of Gothenburg**
**Appendix: Ordered & Multinomial Outcomes. Tobit regression.**
Måns Söderbom

## 1. Ordered probit: Incidence of corruption among Kenyan manufacturing firms

In the following example we consider a model of corruption in the Kenyan manufacturing sector.[1] Our dataset consists of 155 firms observed in year 2000.

Our basic latent model of corruption is

$$corrupt_i^* = \alpha_1 \ln K_i + \alpha_2 \left( \frac{profit}{K} \right)_i + s_i + town_i + e_i \,,$$

where
*corrupt* = incidence of corruption in the process of getting connected to public services
$K$ = Value of the firm's capital stock
*profit* = Total profit
*s* = sector effect (food, wood, textile; metal is the omitted base category)
*town* = location effect (Nairobi, Mombasa, Nakuru; Eldoret – which is the most remote town – is the omitted base category)
*u* = a residual, assumed homoskedastic and normally distributed with variance normalized to one.

Incidence of corruption is not directly observed. Instead we have subjective data, collected through interviews with the firm's management, on the prevalence of corruption. Specifically, each firm was asked the following question:

"Do firms like yours typically need to make extra, unofficial payments to get connected to public services (e.g. electricity, telephone etc)?"

Answers were coded using the following scale:

| N/A | Always | Usually | Frequently | Sometimes | Seldom | Never |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

Observations for which the answer is N/A or missing have been deleted from the data. Notice that this variable, denoted *obribe*, is ordered so that high values indicate relatively low levels of corruption.

Given the data available, it makes sense to estimate the model using either ordered probit or ordered logit.

[1] These data was collected by a team from the CSAE in 2000 – for details on the survey and the data, see Söderbom, Måns "Constraints and Opportunities in Kenyan Manufacturing: Report on the Kenyan Manufacturing Enterprise Survey 2000," 2001, CSAE Report REP/2001-03. Oxford: Centre for the Study of African Economies, Department of Economics, University of Oxford. Available at http://www.economics.ox.ac.uk/CSAEadmin/reports.

Summary statistics for these variables are as follows:

| Variable | Obs | Mean | Std. Dev. | Min | Max |
|---|---|---|---|---|---|
| obribe | 155 | 3.154839 | 1.852138 | 1 | 6 |
| lk | 155 | 15.67499 | 3.197098 | 7.258711 | 22.38821 |
| profk | 155 | -.3647645 | 2.449862 | -15.73723 | 11.3445 |
| wood | 155 | .2 | .4012966 | 0 | 1 |
| textile | 155 | .2903226 | .4553826 | 0 | 1 |
| metal | 155 | .2516129 | .4353465 | 0 | 1 |
| nairobi | 155 | .5096774 | .5015268 | 0 | 1 |
| mombasa | 155 | .2645161 | .442505 | 0 | 1 |
| nakuru | 155 | .1032258 | .3052398 | 0 | 1 |

**Table 1. Ordered probit results**

```
.  oprobit obribe1 lk profk sec2-sec4 nairobi mombasa nakuru

Iteration 0:   log likelihood = -257.79967
Iteration 1:   log likelihood = -248.35111
Iteration 2:   log likelihood = -248.34599
Iteration 3:   log likelihood = -248.34599

Ordered probit estimates                          Number of obs   =        155
                                                  LR chi2(8)      =      18.91
                                                  Prob > chi2     =     0.0154
Log likelihood = -248.34599                       Pseudo R2       =     0.0367
```

| obribe1 | Coef. | Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| lk | -.0809392 | .0307831 | -2.63 | 0.009 | -.141273 | -.0206054 |
| profk | -.0569773 | .0377651 | -1.51 | 0.131 | -.1309955 | .0170409 |
| wood | -.543739 | .2698032 | -2.02 | 0.044 | -1.072543 | -.0149345 |
| textile | .1068028 | .2405553 | 0.44 | 0.657 | -.3646768 | .5782825 |
| metal | -.3959804 | .251102 | -1.58 | 0.115 | -.8881313 | .0961706 |
| nairobi | .0740607 | .2836262 | 0.26 | 0.794 | -.4818364 | .6299578 |
| mombasa | -.1443718 | .3005436 | -0.48 | 0.631 | -.7334265 | .4446829 |
| nakuru | -.0242636 | .3644382 | -0.07 | 0.947 | -.7385494 | .6900222 |
| _cut1 | -2.065609 | .5583871 | (Ancillary parameters) | | | |
| _cut2 | -1.539941 | .5510676 | | | | |
| _cut3 | -1.309679 | .5479021 | | | | |
| _cut4 | -.665663 | .543653 | | | | |
| _cut5 | -.5036779 | .5442082 | | | | |

***Marginal effects***:

```
. mfx compute, predict(outcome(1));

Marginal effects after oprobit
      y  = Pr(obribe1==1) (predict, outcome(1))
         =  .26194813
------------------------------------------------------------------------------
variable |      dy/dx    Std. Err.     z    P>|z|  [    95% C.I.   ]      X
---------+--------------------------------------------------------------------
      lk |   .0263548      .01007    2.62   0.009   .006619   .04609    15.675
   profk |   .0185525      .01232    1.51   0.132  -.005599  .042704  -.364765
    sec2*|   .1920139      .10092    1.90   0.057  -.005785  .389813        .2
    sec3*|  -.0342657      .07595   -0.45   0.652  -.183127  .114596   .290323
    sec4*|   .1359843      .09025    1.51   0.132  -.040909  .312877   .251613
 nairobi*|  -.0241228      .09275   -0.26   0.795  -.205901  .157656   .509677
 mombasa*|   .0479847      .10129    0.47   0.636  -.150547  .246517   .264516
  nakuru*|   .0079487      .12011    0.07   0.947   -.22747  .243368   .103226
------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1

. mfx compute, predict(outcome(3));

Marginal effects after oprobit
      y  = Pr(obribe1==3) (predict, outcome(3))
         =  .09165806
------------------------------------------------------------------------------
variable |      dy/dx    Std. Err.     z    P>|z|  [    95% C.I.   ]      X
---------+--------------------------------------------------------------------
      lk |  -.0000255      .00073   -0.03   0.972  -.001459  .001408    15.675
   profk |  -.0000179      .00051   -0.03   0.972  -.001027  .000991  -.364765
    sec2*|  -.0078598      .00883   -0.89   0.374  -.025173  .009453        .2
    sec3*|  -.0001844      .00132   -0.14   0.889  -.002777  .002408   .290323
    sec4*|  -.0035893      .00571   -0.63   0.529  -.014771  .007593   .251613
 nairobi*|   .0000281      .00068    0.04   0.967  -.001302  .001358   .509677
 mombasa*|  -.0004917      .00236   -0.21   0.835  -.005126  .004143   .264516
  nakuru*|  -.0000289      .00079   -0.04   0.971   -.00158  .001522   .103226
------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1

. mfx compute, predict(outcome(6));

Marginal effects after oprobit
      y  = Pr(obribe1==6) (predict, outcome(6))
         =  .17759222
------------------------------------------------------------------------------
variable |      dy/dx    Std. Err.     z    P>|z|  [    95% C.I.   ]      X
---------+--------------------------------------------------------------------
      lk |  -.0210592      .00817   -2.58   0.010   -.03708 -.005038    15.675
   profk |  -.0148246       .0099   -1.50   0.134  -.034227  .004578  -.364765
    sec2*|  -.1203152        .051   -2.36   0.018  -.220279 -.020351        .2
    sec3*|   .0283602      .06519    0.44   0.664  -.099409  .156129   .290323
    sec4*|  -.0936442      .05426   -1.73   0.084  -.199983  .012695   .251613
 nairobi*|   .0192561      .07374    0.26   0.794   -.12528  .163792   .509677
 mombasa*|  -.0363774      .07328   -0.50   0.620  -.179994  .107239   .264516
  nakuru*|  -.0062568      .09313   -0.07   0.946  -.188796  .176283   .103226
------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1
```

Note: The sign of the marginal effects referring to the highest outcome are the same as the sign of the estimated parameter beta(j), and the sign of the marginal effects referring to the lowest outcome are the opposite to the sign of the estimated parameter beta(j). For intermediate outcome categories, the signs of the marginal effects are ambiguous and often close to zero (e.g. outcome 3 above). Why is this?

## 2. Multinomial Logit

In the following example we consider a model of occupational choice within the Kenyan manufacturing sector (see footnote 1 for a reference for the data). We have data on 950 individuals and we want to investigate if education, gender and parental background determine occupation.

We distinguish between four classes of jobs:

- management
- administration and supervision
- sales and support staff
- production workers

Sample proportions for these four categories are as follows:

```
. tabulate job

        job |      Freq.     Percent        Cum.
------------+-----------------------------------
       Prod |        545       57.37       57.37
      Manag |         91        9.58       66.95
      Admin |        270       28.42       95.37
    Support |         44        4.63      100.00
------------+-----------------------------------
      Total |        950      100.00
```

The explanatory variables are

years of education: educ
gender: male
parental background: f_prof, m_prof (father/mother professional), f_se, m_se (father/mother self-employed or trader)

Summary statistics for these variables are as follows:

```
. sum educ male f_prof f_se m_prof m_se;

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        educ |       950    9.933684     2.86228          0         17
        male |       950    .8136842    .3895664          0          1
      f_prof |       950    .1347368    .3416221          0          1
        f_se |       950    .1231579    .3287915          0          1
      m_prof |       950    .0578947    .2336673          0          1
-------------+--------------------------------------------------------
        m_se |       950    .1315789    .3382105          0          1
```

A breakdown by occupation is a useful first step to see if there are any broad patterns in the data:

```
. tabstat educ male f_prof f_se m_prof m_se, by(job);

Summary statistics: mean
  by categories of: job

     job |      educ      male    f_prof      f_se    m_prof      m_se
---------+------------------------------------------------------------
    Prod |  8.946789  .8825688  .0715596  .1229358  .0348624  .1559633
   Manag |  12.82418  .8021978  .3406593  .1208791  .1318681  .0879121
   Admin |  10.75926  .7037037  .1814815  .1074074  .0666667        .1
 Support |  11.11364  .6590909  .2045455  .2272727  .1363636  .1136364
---------+------------------------------------------------------------
   Total |  9.933684  .8136842  .1347368  .1231579  .0578947  .1315789
----------------------------------------------------------------------
```

The multinomial logit seems a suitable model for modelling occupational choice with these data (notice in particular that there is no natural ordering of the dependent variable).

I begin by coding the job variable from 0 to 3:

job: 0 = prod; 1 = manag; 2 = admin; 3 = support

So I will obtain three vectors of parameter estimates. Because I have set job = 0 for production workers, this will be the **base category** (I can alter this by using the *basecategory( )* option).

Results:

```
. mlogit job educ male f_prof f_se m_prof m_se;

Multinomial logistic regression                   Number of obs   =        950
                                                  LR chi2(18)     =     289.97
                                                  Prob > chi2     =     0.0000
Log likelihood = -846.16161                       Pseudo R2       =     0.1463

------------------------------------------------------------------------------
         job |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Manag        |
        educ |    .738846   .0755869     9.77   0.000     .5906984    .8869935
        male |   .0277387   .3383262     0.08   0.935    -.6353685     .690846
      f_prof |   1.135737   .3373116     3.37   0.001     .4746187    1.796856
        f_se |   .1189543   .4074929     0.29   0.770     -.679717    .9176256
      m_prof |   .3806786   .4661837     0.82   0.414    -.5330247    1.294382
        m_se |  -.6073577   .4413568    -1.38   0.169    -1.472401    .2576856
       _cons |  -10.25324   .9913425   -10.34   0.000    -12.19623   -8.310244
-------------+----------------------------------------------------------------
Admin        |
        educ |   .2421636   .0333887     7.25   0.000     .1767229    .3076042
        male |  -.9075081   .2018354    -4.50   0.000    -1.303098    -.511918
      f_prof |   .5696015   .2570499     2.22   0.027      .065793     1.07341
        f_se |  -.0884656   .2616688    -0.34   0.735     -.601327    .4243958
      m_prof |  -.0135092   .3751632    -0.04   0.971    -.7488156    .7217972
        m_se |  -.5700617    .256966    -2.22   0.027    -1.073706   -.0664175
       _cons |  -2.350944   .3941898    -5.96   0.000    -3.123542   -1.578346
-------------+----------------------------------------------------------------
Support      |
        educ |   .2805316   .0723475     3.88   0.000     .1387331    .4223302
        male |  -.9905816   .3642871    -2.72   0.007    -1.704571    -.276592
      f_prof |   .6547286   .4707312     1.39   0.164    -.2678877    1.577345
        f_se |   .8717071   .4237441     2.06   0.040     .0411839     1.70223
      m_prof |   .7996763   .5500412     1.45   0.146    -.2783846    1.877737
        m_se |  -.5924061   .5213599    -1.14   0.256    -1.614253    .4294405
       _cons |  -4.777905   .8675103    -5.51   0.000    -6.478193   -3.077616
------------------------------------------------------------------------------
(Outcome job==Prod is the comparison group)
```

**Marginal effects**

```
. mfx compute, predict(outcome(1)) nose;

Marginal effects after mlogit
      y  = Pr(job==1) (predict, outcome(1))
         =  .03792548
-------------------------------------------------------------------------------
                        variable |          dy/dx                 X
---------------------------------+---------------------------------------------
                            educ |       .0236809           9.93368
                            male*|       .0136163           .813684
                          f_prof*|       .0448291           .134737
                            f_se*|       .0033605           .123158
                          m_prof*|       .0139252           .057895
                            m_se*|      -.0134499           .131579
-------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1
```

```
. mfx compute, predict(outcome(2)) nose;

Marginal effects after mlogit
      y  = Pr(job==2) (predict, outcome(2))
         =  .30591218
-------------------------------------------------------------------------------
                        variable |          dy/dx                 X
---------------------------------+---------------------------------------------
                            educ |        .0390723             9.93368
                            male*|       -.1879454             .813684
                          f_prof*|        .0950757             .134737
                            f_se*|       -.0354302             .123158
                          m_prof*|       -.0225145             .057895
                            m_se*|       -.0999013             .131579
-------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1

. mfx compute, predict(outcome(3)) nose;

Marginal effects after mlogit
      y  = Pr(job==3) (predict, outcome(3))
         =  .04398022
-------------------------------------------------------------------------------
                        variable |          dy/dx                 X
---------------------------------+---------------------------------------------
                            educ |        .0073048             9.93368
                            male*|       -.0322613             .813684
                          f_prof*|        .0184086             .134737
                            f_se*|        .0519705             .123158
                          m_prof*|        .0458739             .057895
                            m_se*|        -.015106             .131579
-------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1

. mfx compute, predict(outcome(0)) nose;

Marginal effects after mlogit
      y  = Pr(job==0) (predict, outcome(0))
         =  .61218213
-------------------------------------------------------------------------------
                        variable |          dy/dx                 X
---------------------------------+---------------------------------------------
                            educ |       -.0700579             9.93368
                            male*|        .2065904             .813684
                          f_prof*|       -.1583134             .134737
                            f_se*|       -.0199008             .123158
                          m_prof*|       -.0372846             .057895
                            m_se*|        .1284572             .131579
-------------------------------------------------------------------------------
(*) dy/dx is for discrete change of dummy variable from 0 to 1
```

**Predicted job probabilities**

**Education = PRIMARY, SECONDARY, and UNIVERSITY**

```
1. PRIMARY
. list pp1 pp2 pp3 pp0;

     +-------------------------------------------+
     |      pp1        pp2        pp3        pp0 |
     |-------------------------------------------|
  1. | .0108399   .2284537   .0304956   .7302108 |
     +-------------------------------------------+

2. SECONDARY
. list sp1 sp2 sp3 sp0;

     +-------------------------------------------+
     |      sp1        sp2        sp3        sp0 |
     |-------------------------------------------|
  1. | .1274372   .3683361   .0573239   .4469028 |
     +-------------------------------------------+

3. UNIVERSITY

. list up1 up2 up3 up0;

     +-----------------------------------------+
     |      up1       up2        up3       up0 |
     |-----------------------------------------|
  1. | .6057396   .240109   .0435665   .110585 |
     +-----------------------------------------+

Note: 1 = manag; 2 = admin; 3 = support; 0 = prod
```

How these probabilities were calculated:

```
/* first collapse the data: this gives a new data set consisting of one
observations and the sample means of the variables */

. collapse educ male f_prof f_se m_prof m_se;

/* now vary education: since 1985 the Kenyan education system has involved 8 years
for primary education, 4 years for secondary, and 4 years for university */

/* first do primary */
. replace educ = 8;
(1 real change made)

/* get the predicted probability that the ‘individual’ is a manager */
> predict pp1, outcome(1);
(option p assumed; predicted probability)

/* get the predicted probability that the ‘individual’ is admin */
. predict pp2, outcome(2);
(option p assumed; predicted probability)

. predict pp3, outcome(3);
(option p assumed; predicted probability)

. predict pp0, outcome(0);
(option p assumed; predicted probability)

/* now do secondary */
. replace educ=12;
(1 real change made)
```

```
>
> predict sp1, outcome(1);
(option p assumed; predicted probability)

. predict sp2, outcome(2);
(option p assumed; predicted probability)

. predict sp3, outcome(3);
(option p assumed; predicted probability)

. predict sp0, outcome(0);
(option p assumed; predicted probability)

/* finally do university */
. replace educ=16;
(1 real change made)


> predict up1, outcome(1);
(option p assumed; predicted probability)

. predict up2, outcome(2);
(option p assumed; predicted probability)

. predict up3, outcome(3);
(option p assumed; predicted probability)

. predict up0, outcome(0);
(option p assumed; predicted probability)
```

## 3. Illustration: The Hausman test for IIA in multinomial logit is totally useless

```
. use http://www.stata-press.com/data/r9/sysdsn3, clear

(Health insurance data)

.
. /* The results shown in the manual */
. mlogit insure male age

Iteration 0:   log likelihood = -555.85446
Iteration 1:   log likelihood = -551.32973
Iteration 2:   log likelihood = -551.32802

Multinomial logistic regression                   Number of obs   =        615
                                                  LR chi2(4)      =       9.05
                                                  Prob > chi2     =     0.0598
Log likelihood = -551.32802                       Pseudo R2       =     0.0081

------------------------------------------------------------------------------
      insure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Prepaid      |
        male |   .5095747   .1977893     2.58   0.010     .1219148    .8972345
         age |  -.0100251   .0060181    -1.67   0.096    -.0218204    .0017702
       _cons |   .2633838   .2787574     0.94   0.345    -.2829708    .8097383
-------------+----------------------------------------------------------------
Uninsure     |
        male |   .4748547   .3618446     1.31   0.189    -.2343477    1.184057
         age |  -.0051925   .0113821    -0.46   0.648     -.027501     .017116
       _cons |  -1.756843   .5309591    -3.31   0.001    -2.797504   -.7161824
------------------------------------------------------------------------------
(insure==Indemnity is the base outcome)

. estimates store allcats

.
. mlogit insure male age if insure !="Uninsure":insure

Iteration 0:   log likelihood =  -394.8693
Iteration 1:   log likelihood =  -390.4871
Iteration 2:   log likelihood = -390.48643

Multinomial logistic regression                   Number of obs   =        570
                                                  LR chi2(2)      =       8.77
                                                  Prob > chi2     =     0.0125
Log likelihood = -390.48643                       Pseudo R2       =     0.0111

------------------------------------------------------------------------------
      insure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Prepaid      |
        male |   .5144003   .1981735     2.60   0.009     .1259875    .9028132
         age |  -.0101521   .0060049    -1.69   0.091    -.0219214    .0016173
       _cons |   .2678043   .2775562     0.96   0.335    -.2761959    .8118046
------------------------------------------------------------------------------
(insure==Indemnity is the base outcome)

.
```

```
. hausman . allcats, alleqs constant

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |       .         allcats       Difference          S.E.
-------------+----------------------------------------------------------------
        male |    .5144003     .5095747        .0048256         .012334
         age |   -.0101521    -.0100251       -.0001269               .
       _cons |    .2678043     .2633838        .0044205               .
------------------------------------------------------------------------------
                          b = consistent under Ho and Ha; obtained from mlogit
           B = inconsistent under Ha, efficient under Ho; obtained from mlogit

    Test:  Ho:  difference in coefficients not systematic

                  chi2(3) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =        0.08
                Prob>chi2 =      0.9944
                (V_b-V_B is not positive definite)

.
. /* confirm that IIA test is nonsense in model with male dummy only */
. mlogit insure male

Iteration 0:   log likelihood = -556.59502
Iteration 1:   log likelihood = -553.40794
Iteration 2:   log likelihood = -553.40712

Multinomial logistic regression                   Number of obs   =        616
                                                  LR chi2(2)      =       6.38
                                                  Prob > chi2     =     0.0413
Log likelihood = -553.40712                       Pseudo R2       =     0.0057

------------------------------------------------------------------------------
      insure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Prepaid      |
        male |    .477311   .1959282     2.44   0.015     .0932987    .8613234
       _cons |  -.1772065   .0968274    -1.83   0.067    -.3669847    .0125718
-------------+----------------------------------------------------------------
Uninsure     |
        male |     .46019   .3593228     1.28   0.200    -.2440698     1.16445
       _cons |  -1.989585   .1884768   -10.56   0.000    -2.358993   -1.620177
------------------------------------------------------------------------------
(insure==Indemnity is the base outcome)

. estimates store allcats

.
. mlogit insure male if insure !="Uninsure":insure

Iteration 0:   log likelihood = -395.53394
Iteration 1:   log likelihood = -392.53619
Iteration 2:   log likelihood = -392.53611

Multinomial logistic regression                   Number of obs   =        571
                                                  LR chi2(1)      =       6.00
                                                  Prob > chi2     =     0.0143
Log likelihood = -392.53611                       Pseudo R2       =     0.0076

------------------------------------------------------------------------------
      insure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Prepaid      |
        male |    .477311   .1959283     2.44   0.015     .0932987    .8613234
       _cons |  -.1772065   .0968274    -1.83   0.067    -.3669847    .0125718
------------------------------------------------------------------------------
(insure==Indemnity is the base outcome)
```

```
.
. hausman . allcats, alleqs constant

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |       .         allcats       Difference          S.E.
-------------+----------------------------------------------------------------
        male |     .477311      .477311        2.63e-13         .000109
       _cons |   -.1772065    -.1772065       -3.66e-15               .
------------------------------------------------------------------------------
                         b = consistent under Ho and Ha; obtained from mlogit
           B = inconsistent under Ha, efficient under Ho; obtained from mlogit

    Test:  Ho:  difference in coefficients not systematic

                  chi2(2) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =        0.00
                Prob>chi2 =      1.0000
                (V_b-V_B is not positive definite)

.
. /* confirm that IIA test is nonsense in model with constant only */
. mlogit insure

Iteration 0:   log likelihood = -556.59502

Multinomial logistic regression                   Number of obs   =        616
                                                  LR chi2(0)      =       0.00
                                                  Prob > chi2     =          .
Log likelihood = -556.59502                       Pseudo R2       =     0.0000

------------------------------------------------------------------------------
      insure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Prepaid      |
       _cons |  -.0595623   .0837345    -0.71   0.477    -.2236789    .1045544
-------------+----------------------------------------------------------------
Uninsure     |
       _cons |  -1.876917   .1600737   -11.73   0.000    -2.190656   -1.563179
------------------------------------------------------------------------------
(insure==Indemnity is the base outcome)

. estimates store allcats

.
. mlogit insure if insure !="Uninsure":insure

Iteration 0:   log likelihood = -395.53394

Multinomial logistic regression                   Number of obs   =        571
                                                  LR chi2(0)      =       0.00
                                                  Prob > chi2     =          .
Log likelihood = -395.53394                       Pseudo R2       =     0.0000

------------------------------------------------------------------------------
      insure |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Prepaid      |
       _cons |  -.0595623   .0837345    -0.71   0.477    -.2236789    .1045544
------------------------------------------------------------------------------
(insure==Indemnity is the base outcome)

.
```

```
. hausman . allcats, alleqs constant

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |       .         allcats       Difference          S.E.
-------------+----------------------------------------------------------------
       _cons |   -.0595623    -.0595623        7.69e-15               .
------------------------------------------------------------------------------
                         b = consistent under Ho and Ha; obtained from mlogit
          B = inconsistent under Ha, efficient under Ho; obtained from mlogit

    Test:  Ho:  difference in coefficients not systematic

                  chi2(1) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =    -0.00    chi2<0 ==> model fitted on these
                                        data fails to meet the asymptotic
                                        assumptions of the Hausman test;
                                        see suest for a generalized test
```

# 4. Tobit

**Figure 1. The inverse Mills ratio function**

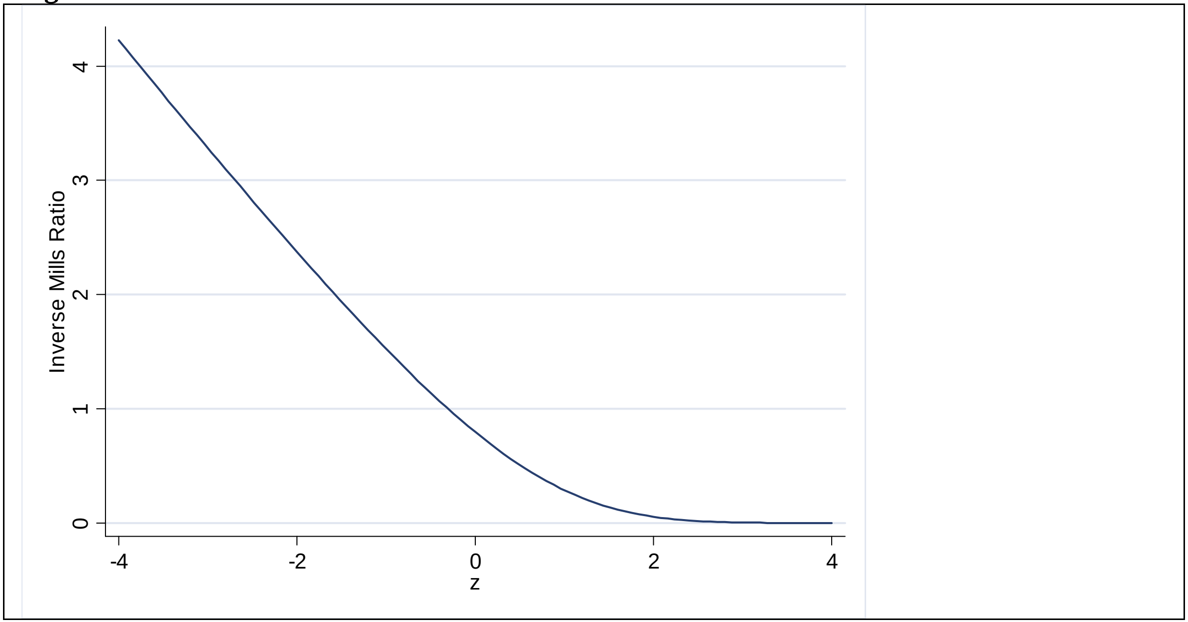



**Illustration: Modelling investment among Ghanaian manufacturing firms**

In the following example we consider a model of company investment within the Ghanaian manufacturing sector.[2] Our dataset consists of 1,202 observations on firms over the 1991-99 period (in fact, there is a panel dimension in the data, but we will ignore this for now).

Our simple model of investment is

$$\left(\frac{I}{K}\right)_{it} = \max\{0, \alpha_0 + \alpha_1 \ln TFP_{it} + \alpha_2 \ln K_{i,t-1} + u_{it}\},$$

where

> *I* = Gross investment in fixed capital (plant & machinery)
> *K* = Value of the capital stock
> *TFP* = Total factor productivity, defined as ln(output) – 0.3ln(K) – 0.7ln(L), where L is employment
> *u* = a residual, assumed homoskedastic and normally distributed.

There is evidence physical capital is 'irreversible' in African manufacturing, i.e. selling off fixed capital is difficult due to the lack of a market for second hand capital goods (Bigsten et al., 2005). We can thus view investment as a corner response variable: investment is bounded below at zero.

Summary statistics for these variables are as follows:

[2] This is an extension of the dataset used by Söderbom and Teal (2004).

**Table 1. Summary statistics**

| Variable | Obs | Mean | Std. Dev. | Min | Max |
|---|---|---|---|---|---|
| invrate | 1202 | .0629597 | .1477861 | 0 | 1 |
| invdum | 1202 | .4550749 | .4981849 | 0 | 1 |
| tfp | 1202 | 10.20903 | 1.108122 | 5.049412 | 14.7326 |
| lk_1 | 1202 | 16.06473 | 3.104121 | 9.555573 | 23.51505 |

Note: invrate = (I/K); invdum = 1 if invrate>0, = 0 if invrate=0; tfp = ln(TFP);
lk_1 = ln[K(t-1)]

**Table 2. OLS results**

```
. reg invrate tfp lk_1;
```

| Source | SS | df | MS |
|---|---|---|---|
| Model | .197262981 | 2 | .098631491 |
| Residual | 26.0334412 | 1199 | .021712628 |
| Total | 26.2307042 | 1201 | .02184072 |

| | | |
|---|---|---|
| Number of obs | = | 1202 |
| F( 2, 1199) | = | 4.54 |
| Prob > F | = | 0.0108 |
| R-squared | = | 0.0075 |
| Adj R-squared | = | 0.0059 |
| Root MSE | = | .14735 |

| invrate | Coef. | Std. Err. | t | P>\|t\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| tfp | .0114908 | .0038443 | 2.99 | 0.003 | .0039484 | .0190331 |
| lk_1 | .0002798 | .0013724 | 0.20 | 0.838 | -.0024127 | .0029723 |
| _cons | -.058845 | .0440225 | -1.34 | 0.182 | -.1452148 | .0275248 |

**Table 3. Tobit results**

```
. tobit invrate tfp lk_1, ll(0);
```

Tobit estimates

Log likelihood = -398.5866

| | | |
|---|---|---|
| Number of obs | = | 1202 |
| LR chi2(2) | = | 44.34 |
| Prob > chi2 | = | 0.0000 |
| Pseudo R2 | = | 0.0527 |

| invrate | Coef. | Std. Err. | t | P>\|t\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| tfp | .0344135 | .0077922 | 4.42 | 0.000 | .0191257 | .0497012 |
| lk_1 | .0123672 | .0027384 | 4.52 | 0.000 | .0069947 | .0177397 |
| _cons | -.6158372 | .0913444 | -6.74 | 0.000 | -.7950496 | -.4366247 |
| _se | .2540915 | .0083427 | (Ancillary parameter) | | | |

Obs. summary: 655 left-censored observations at invrate<=0
547 uncensored observations

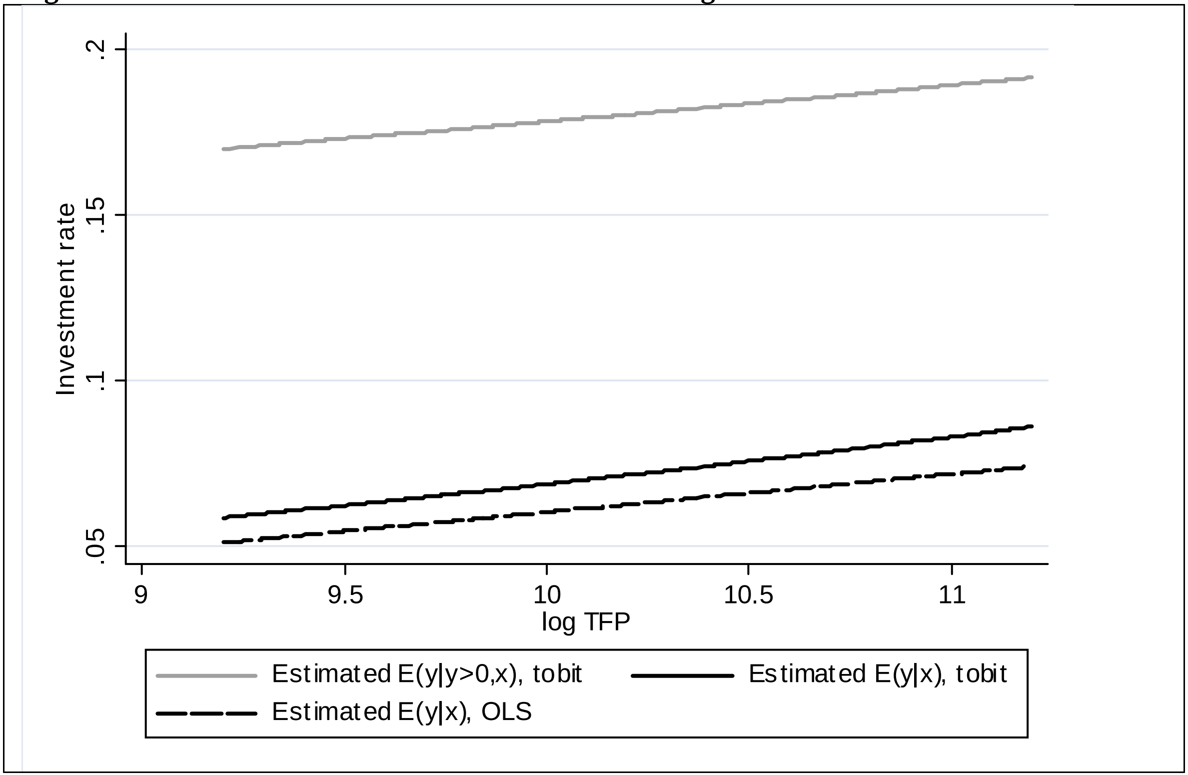
*Marginal effects based on tobit*

```
. mfx compute, predict(e(0,.));
```

Marginal effects after tobit
y = E(invrate|invrate>0) (predict, e(0,.))
= .18058807

| variable | dy/dx | Std. Err. | z | P>\|z\| | [ 95% C.I. | ] | X |
|---|---|---|---|---|---|---|---|
| tfp | .0106934 | .00241 | 4.44 | 0.000 | .005969 | .015418 | 10.209 |
| lk_1 | .0038429 | .00084 | 4.56 | 0.000 | .00219 | .005496 | 16.0647 |

**Figure 2. Predicted investment rates as a function of log TFP**




Note: Evaluated at the sample mean of lk_1.

**References**

Bigsten, Arne, Paul Collier, Stefan Dercon, Marcel Fafchamps, Bernard Gauthier, Jan Willem Gunning, Remco Oostendorp, Catherine Pattillo, Måns Söderbom, and Francis Teal (2005). "Adjustment Costs, Irreversibility and Investment Patterns in African Manufacturing," *The B.E. Journals in Economic Analysis & Policy: Contributions to Economic Analysis & Policy* 4:1, Article 12, pp. 1-27.

Söderbom, Måns, and Francis Teal (2004). "Size and Efficiency in African Manufacturing Firms: Evidence from Firm-Level Panel Data," *Journal of Development Economics* 73, pp. 369-394.