

## Appendix for Bond and Söderbom (2009): The dynamic programming model

This appendix describes the numerical procedure used to generate the simulated investment data analysed in section 4 of Bond and Söderbom, “Conditional investment-cash flow sensitivities and financing constraints”, August 2009.

### The model

The firm’s objective is to maximize the value of the firm:

$$\begin{aligned} V(K_t, B_{t-1}; a_t^P, a_t^T, e_t) = & \max_{I_t, N_t, B_t} \{ \Pi(K_t, I_t; a_t^P, a_t^T, e_t) - \Phi(K_t, N_t) \\ & + B_t - (1 + i(K_t, B_{t-1}))B_{t-1} \\ & + \beta E_t [V((1 - \delta)K_t + I_t, B_t; a_{t+1}^P, a_{t+1}^T, e_{t+1})] \}, \end{aligned}$$

subject to the following constraints (which hold for all  $t$ ):

(i) Non-negative dividends:

$$\Pi(K_t, I_t; a_t^P, a_t^T, e_t) - \Phi(K_t, N_t) + B_t - (1 + i(K_t, B_{t-1}))B_{t-1} + N_t \geq 0,$$

(ii) Non-negative new equity:

$$N_t \geq 0$$

(iii) Non-negative debt:

$$B_t \geq 0$$

(iv) Capital evolution:

$$K_{t+1} = I_t + (1 - \delta) K_t$$

(v) Non-negative capital:

$$K_{t+1} \geq 0.$$

## Exploiting linear homogeneity of the value function

Since the value function is linear homogeneous in capital, we can eliminate  $K_t$  from the state space by normalizing by  $K_t$  (see, for example, Bloom, 2009). Using the interest rate schedule in eq. (20),

$$i(K_t, B_{t-1}) = i + \eta \left( \frac{B_{t-1}}{K_t} \right),$$

the capital evolution constraint,

$$\frac{K_{t+1}}{K_t} = I_t/K_t - \delta + 1,$$

and defining  $\pi_t = \Pi_t/K_t$ ,  $v_t = V_t/K_t$ ,  $\phi_t = \Phi_t/K_t$ ,  $b_{t-1} = B_{t-1}/K_t$ ,  $n_t = N_t/K_t$ , and

$$\Psi_t = [I_t/K_t - \delta + 1] b_t - [1 + i + \eta b_{t-1}] b_{t-1}.$$

we can write the value of the firm as

$$\begin{aligned} v(b_{t-1}; a_t^P, a_t^T, e_t) = & \max_{I_t/K_t, n_t, b_t} \{ \pi(I_t/K_t; a_t^P, a_t^T, e_t) - \phi(n_t) \\ & + \Psi_t + [I_t/K_t - \delta + 1] \beta E_t [v(b_t; a_{t+1}^P, a_{t+1}^T, e_{t+1})] \}, \end{aligned} \quad (\text{A1})$$

subject to the following constraints (which hold for all  $t$ ):

(i) Non-negative dividends:

$$\pi(I_t/K_t; a_t^P, a_t^T, e_t) - \phi(n_t) + \Psi_t + n_t \geq 0$$

(ii) Non-negative new equity:

$$n_t \geq 0$$

(iii) Non-negative debt:

$$b_t \geq 0$$

(iv) Non-negative capital:

$$I_t/K_t - \delta + 1 \geq 0.$$

## Solving the Bellman equation using value function iteration

We solve the firm's optimization problem using value function iteration. The principles of our algorithm are as follows.

1. Start with a guess for the true value function  $v(b_{t-1}; a_t^P, a_t^T, e_t)$ . Call this guess  $v^1$ . Use it on the right-hand side of the Bellman equation (A1), and compute  $E_t [v(b_t; a_{t+1}^P, a_{t+1}^T, e_{t+1})]$  (more on how to compute this expectation below). Find the values of  $\{I_t/K_t, n_t, b_t\}$  that maximize firm value subject to the relevant constraints.
2. Update the guess for the true value function  $v(b_{t-1}; a_t^P, a_t^T, e_t)$  using the solution obtained in the previous step (i.e. this is the value  $v$  associated with optimal  $\{I_t/K_t, n_t, b_t\}$ ). Call this updated guess  $v^2$ . Check if  $v^2 = v^1$ ; if true, we have converged to the true function and so iteration can stop; if not set  $j = 2$  and go to step 3.
3. Set  $j = j + 1$ . Use  $v^{j-1}$  on the right-hand side of the Bellman equation to compute  $E_t [v(b_t; a_{t+1}^P, a_{t+1}^T, e_{t+1})]$ , and calculate the optimal choice rule  $\{I_t/K_t, n_t, b_t\}$ , subject to the relevant constraints. Update the value function,  $v^j$ . Check if  $v^j = v^{j-1}$ ; if true, there is convergence and so iteration stops; if not, repeat step 3.

To implement this method we need to discretize the state and control space, and we need a way of calculating the expected value of the firm in  $t + 1$ . We turn to these issues next.

### State space and control space

We discretize the distribution of the debt variables  $b_{t-1}$  (state) and  $b_t$  (control) by constructing a finite set of permissible values ( $bk_1, bk_2, \dots, bk_{M_b}$ ) where  $bk_1$  is the lowest permissible value,  $bk_{M_b}$  the highest, and  $M_b$  is the number of permissible values. The results in the paper are based on models in which  $bk_1 = 0$ , imposing non-negative borrowing (thus  $bk_1$  is determined by an economic constraint). The upper limit is set sufficiently high not to bind. Throughout we use  $M_b = 18$ .

We discretize the distribution of the investment variable ( $I_t/K_t$ ) (control) by constructing a finite set of permissible values ( $ik_1, ik_2, \dots, ik_{M_i}$ ) where  $ik_1$  is the lowest permissible value,  $ik_{M_i}$  the highest, and  $M_i$  is the number of permissible values. The upper and lower limits are set such that they will not bind. Throughout we use  $M_i = 120$ .

The distribution of new equity  $n_t$  is not discretized. Define

$$\Lambda_t = \begin{cases} 0 & \text{if } \pi(I_t/K_t; a_t^P, a_t^T, e_t) + \Psi_t \geq 0 \\ 1 & \text{if } \pi(I_t/K_t; a_t^P, a_t^T, e_t) + \Psi_t < 0 \end{cases}.$$

Thus  $\Lambda_t = 1$  implies that the firm has to issue new equity to satisfy the non-negativity constraint for dividends. With the cost of issuing new equity given by  $\phi(n_t) = 0.5\phi n_t^2$ , optimal new equity  $n_t$  is as follows:

$$n_t = \begin{cases} 0 & \text{if } \Lambda_t = 0 \\ -[\pi(I_t/K_t; a_t^P, a_t^T, e_t) - 0.5\phi n_t^2 + \Psi_t] > 0 & \text{if } \Lambda_t = 1 \end{cases}$$

If  $\phi > 0$ , the firm will be financially constrained whenever  $\Lambda_t = 1$ , in which case new equity satisfies the quadratic

$$n_t + \pi(I_t/K_t; a_t^P, a_t^T, e_t) - 0.5\phi n_t^2 + \Psi_t = 0. \quad (\text{A2})$$

Optimal new equity is the positive root of (A2):

$$n_t = \frac{-1 + \sqrt{1 - 2\phi[\pi(I_t/K_t; a_t^P, a_t^T, e_t) + \Psi_t]}}{\phi},$$

which is always real (since  $\pi(I_t/K_t; a_t^P, a_t^T, e_t) + \Psi_t < 0$ ).

The serially uncorrelated shocks  $a_t^T$  and  $e_t$ , assumed normally distributed in the model, are approximated using a discrete Gauss-Hermite quadrature. This implies the value function is evaluated at a finite set of variance dependent values  $\sigma_Z \sqrt{2} [x_1 \ x_2 \ \dots \ x_{M_Z}]$ , for  $Z = \{T, e\}$ , where the  $x_j$  are values (nodes) determined by the Gauss-Hermite quadrature, and  $M_T, M_e$  denote the number of nodes. The expectation of the value function  $E_t [v(b_t; a_{t+1}^P, a_{t+1}^T, e_{t+1})]$  is computed by summing over the  $M_T \times M_e$  possible realizations of the value function conditional on  $b_t$  and  $a_{t+1}^P$  using a set of weights determined by the Gauss-Hermite quadrature. See Judd (1998), pp.261-263 for details. Throughout we use  $M_T = M_e = 3$ .

The serially correlated component of the productivity parameter  $a^P$  is approximated using the method proposed by Tauchen (1986). The value function is evaluated at a finite set of possible realizations  $(\bar{a}_1^P, \bar{a}_2^P, \dots, \bar{a}_{M_P}^P)$ , where  $M_P$  is the number of possible realizations. A transition matrix  $\Gamma$  is constructed, consisting of elements  $\Gamma_{ij} = \Pr(a_{t+1}^P = \bar{a}_j^P | a_t^P = \bar{a}_i^P)$  determined by the persistence parameter  $\rho$  and the variance parameter  $\sigma_u$ . Having integrated over the distribution of  $a^T$  and  $e$  as described in the previous paragraph, we obtain the expectation  $E_t[v(b_t; a_{t+1}^P, a_{t+1}^T, e_{t+1})]$ , conditional on the policy  $b_t$  and current state  $\bar{a}_i^P$  by summing the value function over the  $M_P$  possible realizations of  $a_{t+1}^P$  using the elements in the  $i$ th column of the transition matrix  $\Gamma$  as weights. Throughout we set  $M_P = 25$ .

### The simulated data

Once the value iteration has converged, we recover the policy functions for investment, debt and new equity. Based on these we simulate panel data according to the principles described in the text.

## References

- [1] Bloom, N. (2009), “The impact of uncertainty shocks”, *Econometrica*, 77:623-685.
- [2] Judd, K.L. (1998), *Numerical Methods in Economics*, Cambridge: The MIT Press.
- [3] Tauchen, G. (1986), “Finite state Markov-Chain approximations to univariate and vector autoregressions”, *Economics Letters*, 20, 177-181.