Macroeconomics I: Investment

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1 Introduction

- Most economists agree that innovation and accumulation of modern **fixed capital** - plant and equipment - in the private sector are important for sustainable increases in per capita incomes, and the standard of living more generally.

- It is sometimes argued that new investment may generate learning externalities or be the leading channel through which innovations drive growth.

- New technology may also be good for the environment.

- Bottom line: An improved understanding of the determinants of investment will improve our understanding of some key aspects of economic progress.

- This lecture provides:
  - An introduction to **conventional models** of investment. These are **linear models** suitable for regression analysis - e.g. accelerator, Tobin's Q model, Euler equation.
  - An introduction to the **empirical literature** based on conventional models that studies the effects on investment of
    * financial constraints; and
    * uncertainty
  - An introduction to the **new** investment literature, which typically uses a structural approach and goes beyond regression analysis when estimating parameters of interest.

About the theoretical modeling

- **Micro to macro.** Today, most macro papers on investment build their model at the level of the firm. So I will stress models of firm behaviour.

About the applications

- I focus mostly on the effects of **financial constraints** and **uncertainty**.

- The investment literature has made significant advances in these areas over the last 20 years.

- Also, given the current economic climate, understanding uncertainty and financial constraints would seem rather relevant.

About the empirical methods

- Vast majority of empirical studies based on regression analysis

- Several important papers in the most recent literature go beyond regression analysis. Instead, they base the estimation of parameters on **matching moments**: i.e. real moments, obtained from the data, are matched with moments simulated numerically based on a theoretical model.

Literature

- The lecture is based on several papers (see bibliography) and you will not have time to read them all. The current lecture notes are meant to be fairly self-contained - in other words if you know and understand the material in these notes you know enough about investment to pass this part of the macro course.

- Having said that, I would of course encourage you to consult the underlying papers in order to get a deeper understanding of the issues.

- Please read at least the introduction for each of the papers listed in the bibliography.

2 Conventional Investment Models

Reference: Robert Chirinko (1993) "Business Fixed Investment Spending: Modeling Strategies, Empirical Results, and Policy Implications," Journal of Economic Literature 31, pp. 1875-1911.

Many different approaches have been used for analyzing investment. Four key issues arising in such research:

1. Consistency of the theoretical model

2. Characteristics of the technology

3. Treatment of expectations

4. The impact of prices, quantities, and shocks on investment

Points (1)-(3) revolve around model specification, whereas (4) is mainly an empirical question. By the early 1990s (when Chirinko wrote his survey article), many economists argued that the empirical investment literature had basically failed to provide useful and credible answers to important economic questions (e.g. regarding the effect of prices on investment). However I think it is fair to say that, since then, significant progress has been made on the empirical side too.

An important distinction in the theoretical modelling of investment concerns how the **dynamics** are introduced. Early work in the literature derived dynamic investment equations by adding lags, in a rather ad hoc fashion, to an equation based on the static first order condition (f.o.c.) for capital. Such models are referred to by Chirinko as models with implicit dynamics. Let's have a look at this class of models.

2.1 Models with implicit dynamics

Suppose the firm chooses capital in order to maximize profits Assuming that the production function exhibits constant elasticity of substitution between capital and variable inputs (labour, intermediate inputs), the static first-order condition for capital is

$$K_t^* = \alpha Y_t C_t^{-\sigma}, \tag{1}$$

where σ is the elasticity of substitution between capital and , α is a technology parameter and

$$C_t = p_r^I (r_t + \delta)$$

is the user cost of capital (I abstract from various taxes affecting the user cost; see Chirinko). To obtain an investment equation from (1), we distinguish between net investment, I_t^n (changes to the capital stock after depreciation), and replacement investment, I_t^r (the expenditure necessary to prevent the capital stock from diminishing due to depreciation):

$$I_t = I_t^n + I_t^r.$$

- Assume that net investment is determined by a **distributed lag** on new orders:

$$I_t^n = \sum_{j=0}^{J} \beta_j \Delta K_{t-j}^*. \qquad (2)$$

  You might wonder where the lags come from. This expression has no formal theoretical justification, really; rather it is written as a distributed lag in order to capture the fact(?) that it takes some time between the occurrence of a shock to desired capital, ΔK_{t-j}^*, and the ordering or installation of new capital.

- Replacement investment is simply

$$I_t^g = \delta K_{t-1}.$$

- Using these ingredients, we can obtain what Chirinko refers to as the **Neoclassical Model of Investment**:

$$I_t = I_t^n + I_t^r = \delta K_{t-1} + \sum_{j=0}^{J} \beta_j \Delta \left(Y_{t-j} C_{t-j}^{-\sigma} \right) + u_t,$$

where u_t is an error term. This is clearly a dynamic equation, but note that the origins of the dynamics are basically ad hoc.

- **Special case I**: Set $\sigma = 0$ (e.g. Leontief technology) and you get the

flexible accelerator model:

$$I_t = I_t^n + I_t^r = \delta K_{t-1} + \sum_{j=0}^{J} \beta_j \Delta Y_{t-j} + u_t,$$

implying that quantity shocks (think output) impact on investment, whereas shocks to the user cost of capital (e.g. in the form of a reduction in interest rate) will have no effect other than through Y (it's perfectly possible that a reduction in the interest rate raises consumer demand, for example - hence it would be wrong to argue that monetary policies can't impact on investment in this model).

- **Special case II**: Set $\sigma = 0$ (e.g. Leontief technology) and $J = 0$ and you get the **simple accelerator model**:

$$I_t = I_t^n + I_t^r = \delta K_{t-1} + \beta_0 \Delta Y_t + u_t,$$

which is similar to the flexible accelerator model except that an output shock in period t has no direct effect on investment beyond period t.

2.1.1 Critique - revisiting the "four issues"

1. Consistency of the theoretical model

   (a) Output and capital are chosen simultaneously by the firm. It is therefore inappropriate to treat output shocks as exogenous in empirical work based on the accelerator model, for example:

   $$I_t = I_t^n + I_t^r = \delta K_{t-1} + \sum_{j=0}^{J} \beta_j \Delta Y_{t-j} + u_t.$$

   (b) Awkward theoretical inconsistency: Desired capital is derived under the assumption that the delivery of the capital goods is immediate, yet in order to derive the dynamic neoclassical investment equation we have to add distribution lags.

(c) Desired capital may not be defined, for instance under perfect competition and constant returns to scale. Hence to use this model we must assume the profit function is non-homogeneous (strictly concave).

2. Characteristics of the technology

   (a) Vintage effects. You may not be able to alter the way (proportions) other inputs (e.g. labour, intermediate inputs) are combined with capital once the capital stock has been installed ('putty-clay'). Has implications for the dynamics.

   (b) Constant geometric depreciation is dubious.

3. Treatment of expectations

(a) The neoclassical model mis-specified unless firms hold static expectations (= expect everything to always be the same as now). In the presence of non-static expectations and delivery lags, you need to add lags in the shocks to the user cost of capital, and in output shocks, separately (see eq. 6 in Chirinko).

(b) The Lucas Critique: Problematic to evaluate the effects of a policy change based on a non-structural regression model, since policy likely impacts on coefficients in an unknown way.

4. The impact of prices, quantities, and shocks on investment

   (a) No clear-cut empirical answer as to the relative roles of output and prices (ucc) as determinants of investment. Chirinko argues, however, that the evidence is in favour of output - not prices - being the dominant determinant of investment.

Response to this: I think it's fair to say that significant progress has been made with regards to (1) and (3). The points in (2) are (much) less emphasized now than they were in the 1980s and early 1990s. Regarding (4), I'd say this type of question is no longer as central as it used to be in empirical research. Today, the two main questions in empirical research on investment concern the role of financial constraints, and uncertainty.

2.2 Models with explicit dynamics

Following the Lucas critique, theories of investment changed in two fundamental ways.

- Models in which the dynamics were added in an ad hoc way were no longer thought appropriate. Instead, the dynamics should follow from the underlying theory of profit maximization. **Adjustment costs** became an important model ingredient, as a result.

- **Rational expectations**. Firms are assumed to understand, and behave according to, the model written down by the economist. Expectations therefore need to be consistent with the model.

2.2.1 The benchmark model

- **Intertemporal optimization**: Underlying this class of model is the assumption that the firm's objective is to **maximize the value of the firm**, defined as the present discounted value of all (expected) future profit streams:

$$V_t = \max_{L_t, K_t} E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{-(s-t)} \Pi \left(L_s, K_s, I_s; \tau_s \right),$$

subject to the capital evolution constraint

$$K_t = (1-\delta) K_{t-1} + I_t,$$

where V_t defines the value of the firm at time t, r is the one-period (constant) discount rate, Π_t is profits, K_t is physical capital, L_t is labour, I_t is investment, and δ is the constant depreciation rate.

- Assumptions:

  – The firm is a price-taker in input and output markets.

  – Output Y_t is determined by labour L_t, capital K_t and a technology shock τ_t:

  $$Y_t = F\left(L_t, K_t; \tau_t\right),$$

  where $F\left(.\right)$ denotes the production function.

  – The purchase price of capital is denoted p_t^I.

  – Capital is "quasi-fixed", in the sense that changing the capital stock is associated with adjustment costs, represented by $G\left(I_t, K_t; \tau_t\right)$. A very common functional form is the **quadratic** specification

$$G\left(I_t, K_t; \tau_t\right) = \left(\frac{\alpha}{2}\right)\left[\frac{I_t}{K_t} - \tau_t\right]^2 K_t,$$

implying that adjustment costs increase at an increasing rate. Too rapid accumulation of capital is thus very costly. More on this below.

– Labour is perfectly flexible (no adjustment costs) and can be hired at wage rate w_t.

– Capital depreciates at a constant rate δ, so that

$$K_t = (1 - \delta) K_{t-1} + I_t.$$

– The price of output is normalized to 1.

- Under these assumptions, the firm's optimization problem can be expressed as follows:

$$\begin{aligned} V_t &= \max_{L_t, K_t} E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{-(s-t)} \{F(L_t, K_t; \tau_t) - G(I_t, K_t; \tau_t) \\ &\quad -p_t^I I_t - w_t L_t\}, \end{aligned}$$

subject to

$$K_s = (1-\delta) K_{s-1} + I_s.$$

This can be re-written as a Bellman equation:

$$\begin{aligned} V_t &= \max_{L_t, K_t} \{F(L_t, K_t; \tau_t) - G(I_t, K_t; \tau_t) - p_t^I I_t - w_t L_t \\ &\quad +\beta E_t V_{t+1}\}, \end{aligned}$$

where $\beta = (1+r)^{-1}$, subject to the capital evolution constraint. This is not how Chirinko presents the problem, but since this way of proceeding is very common in the current literature I adopt the Bellman equation approach. Based on this maximization problem, optimal labour and capital will satisfy the following conditions:

- Labor:

$$F_L\left(L_t, K_t; \tau_t\right) = w_t,$$

i.e. a standard non-dynamic first-order condition.

- Investment:

$$
\begin{aligned}
G_I(I_t, K_t; \tau_t) + p_t^I &= F_K(L_t, K_t; \tau_t) \\
&\quad - G_K(I_t, K_t; \tau_t) + \beta E_t \frac{\partial V_{t+1}}{\partial K_t} \\
G_I(I_t, K_t; \tau_t) + p_t^I &= F_K(L_t, K_t; \tau_t) \\
&\quad - G_K(I_t, K_t; \tau_t) + \beta (1 - \delta) E_t \frac{\partial V_{t+1}}{\partial K_{t+1}}.
\end{aligned}
$$

- Using the quadratic adjustment cost function

$$
G(I_t, K_t; \tau_t) = \left(\frac{\alpha}{2}\right) \left[\frac{I_t}{K_t} - \tau_t\right]^2 K_t,
$$

we have

$$
G_I(I_t, K_t; \tau_t) = \alpha \left[\frac{I_t}{K_t} - \tau_t\right],
$$

and so we can write the f.o.c. for investment as

$$\alpha\left[\frac{I_t}{K_t}-\tau_t\right]+p_t^I=\frac{\partial V_t}{\partial K_t},$$

where

$$\frac{\partial V_t}{\partial K_t}=F_K\left(L_t,K_t;\tau_t\right)-G_K\left(I_t,K_t;\tau_t\right)+\beta\left(1-\delta\right)E_t\frac{\partial V_{t+1}}{\partial K_{t+1}}$$

denotes the shadow value of capital (the increase in the firm value that would result if the firm were 'given' another unit of physical capital). This gives us the following benchmark model:

$$\frac{I_t}{K_t}=\frac{1}{\alpha}\left(\frac{\partial V_t}{\partial K_t}-p_t^I\right)+\tau_t, \tag{3}$$

where the error term is interpretable as an adjustment cost shock.

- This equation is straightforward to interpret: whenever there is a discrepancy between the shadow value of capital and the unit purchase price, the firm has an incentive to change the capital stock - but its actions are tempered by the adjustment cost parameter α.

- Clearly, the higher is α, the more slowly investment responds to changes in the underlying 'desire' to invest.

- Attractive features of (3):
  - Derived directly from an optimization problem - hence not "ad hoc".
  - Rational expectations
  - Even the error term has a theoretical interpretation (what is it?).

- How can it be used empirically?

- The operational problem is to relate $\frac{\partial V_t}{\partial K_t}$ to observable variables. At the time when Chirinko wrote his paper the two most popular approaches were the q model and the Euler equation approach.

2.2.2 q models

- From an empirical point of view, the benchmark model is not very useful unless the shadow value of capital, $\frac{\partial V_t}{\partial K_t}$, which is often termed **marginal q,** can be expressed in terms of observables.

- Define Tobin's average q as the ratio of the value of the firm V_t to the replacement cost of its existing capital stock:

$$q_t^A = \frac{V_t}{p_t^I K_t}.$$

- Hayashi (1982) showed that

$$V_t = \frac{\partial V_t}{\partial K_t} K_t$$

under the following assumptions:

– Product & factor markets are competitive

– Production and adjustment cost technologies are linear homogeneous (constant returns)

– Capital is homogeneous

– Investment decisions are separate from other real & financial decisions.

- Under these assumptions, we can re-write the benchmark model as follows:

$$\begin{aligned}
\frac{I_t}{K_t} &= \frac{1}{\alpha}\left(\frac{\partial V_t}{\partial K_t} - p_t^I\right) + \tau_t \\
\frac{I_t}{K_t} &= \frac{1}{\alpha}\left(\frac{V_t}{K_t} - p_t^I\right) + \tau_t, \\
\frac{I_t}{K_t} &= \frac{1}{\alpha}\left(q_t^A - 1\right) p_t^I + \tau_t, \\
\frac{I_t}{K_t} &= (1/\alpha)\, q_t + \tau_t,
\end{aligned}$$

where $q_t = \left(q_t^A - 1\right) p_t^I$.

- This is very useful from an empirical point of view, since q_t^A is straightforward to measure; all we need are data on the **value of the firm** (stock market data are often used) and the **replacement value of the capital stock** (available from the balance sheet).

- Equipped with such data the applied researcher can thus run regress investment rates on some measure of q and identify the adjustment cost parameter α (the p_t^I is often suppressed, appealing to constant input prices across firms, or, if the data have a time series dimension, represented by a time trend or time dummies).

- Equipped with an estimate of α we can predict how strongly investment will respond to shocks affecting the firm value (e.g. a cut in interest rates or a positive demand shock).

- Note that the problem of unobservable expectations is solved by equating a forward-looking variable, i.e. the marginal effect of capital on discounted expected future profits, to one that is observable, i.e. the average q.

- Note that average q controls for "everything": conditional on average q, no other variable should determine investment (assuming p_t^I is constant in the cross-section of firms). Hence, average q is said to be a **sufficient statistic** for investment. As we will see below, this is a useful starting point when testing for the effects of financial constraints on investment.

Potential problems

- If **stock market** data are used to determine the value of the firm (which is the most common approach), it is clearly important that the stock market gets the valuation of the firm right. That it does should not be taken for granted - think of share price bubbles for example - in which case marginal q is effectively measured with error.

- Furthermore, the capital stock may be measured with error too. This may lead to bias in the estimate of the adjustment cost parameter.

- Another reason why the average q approach is potentially problematic is that the underlying assumptions appear quite restrictive - especially perfect competition and constant returns to scale. While it may be possible approximate of marginal q under imperfect competition or decreasing returns to scale, it is not - as far as I know - possible to express marginal q in terms of observables exactly.

- And of course adjustment costs may not in fact be quadratic, in which case the model will be mis-specified.

- The q model's **empirical performance** has not been very satisfactory. A key disturbing fact is that estimates of the coefficient on average q typically are **rather low** (less than 0.05 usually) implying very (implausibly) high adjustment costs. Summers (1981), cited on p.1892 in Chirinko, obtains $\alpha = 32$, which implies that 20 years after an unexpected change in the economic environment, the capital stock would have moved only 75% of the way to the new steady-state value.

2.2.3 Euler equations

Our benchmark model (re-arranged; and with u_t replacing τ_t in the adjustment cost function):

$$\alpha\left(\frac{I_t}{K_t} - u\right) + p_t^I = \frac{\partial V_t}{\partial K_t}. \tag{4}$$

We saw in the previous section how, under certain assumptions, can be written as an equation in which investment depends on average q.

- An alternative route open to us is to use the structure of the model to derive the Euler equation for investment. The Euler equation can be derived in different ways; one straightforward approach is as follows:

- First, decompose the shadow value of capital:

$$\frac{\partial V_t}{\partial K_t} = F_K\left(L_t, K_t; \tau_t\right) - G_K\left(I_t, K_t\right) + \beta\left(1-\delta\right) E_t \frac{\partial V_{t+1}}{\partial K_{t+1}}.$$

- Second, write the benchmark model in $t+1$ and take expectations on both sides:

$$\alpha E_t\left(\frac{I_{t+1}}{K_{t+1}} - u\right) + E_t p^I_{t+1} = E_t\left(\frac{\partial V_{t+1}}{\partial K_{t+1}}\right).$$

Multiply by $\beta\left(1-\delta\right)$:

$$\begin{aligned}\beta\left(1-\delta\right) E_t\left(\frac{\partial V_{t+1}}{\partial K_{t+1}}\right) &= \beta\left(1-\delta\right)\alpha E_t\left(\frac{I_{t+1}}{K_{t+1}} - u\right) \\ &\quad +\beta\left(1-\delta\right) E_t p^I_{t+1}.\end{aligned}$$

- Third, use this expression in the decomposition of the shadow value of capital:

$$\begin{aligned}\frac{\partial V_t}{\partial K_t} &= F_K\left(L_t, K_t; \tau_t\right) - G_K\left(I_t, K_t\right) \\ &\quad + \beta\left(1-\delta\right)\alpha E_t\left(\frac{I_{t+1}}{K_{t+1}} - u\right) + \beta\left(1-\delta\right)E_t p_{t+1}^I.\end{aligned}$$

- Fourth, plug this into the benchmark model:

$$\alpha\left(\frac{I_t}{K_t} - u_t\right) + p_t^I = \frac{\partial V_t}{\partial K_t},$$

$$
\begin{aligned}
\alpha\left(\frac{I_t}{K_t}-u\right)+p_t^I &= F_K\left(L_t,K_t;\tau_t\right)-G_K\left(I_t,K_t\right)\\
&\quad+\beta\left(1-\delta\right)\alpha E_t\left(\frac{I_{t+1}}{K_{t+1}}-u\right)\\
&\quad+\beta\left(1-\delta\right)E_t p_{t+1}^I.
\end{aligned}
$$

- Finally, write $X_{t+1}=E_t\left[X_{t+1}\right]+\epsilon_{t+1}$, where ϵ_{t+1} denotes a forecast error, and use the functional form of the adjustment cost function to parameterize $G_K\left(I_t,K_t\right)$:

$$
\begin{aligned}
\left(\frac{I_t}{K_t}\right) &= cons+\beta\left(1-\delta\right)\left(\frac{I_{t+1}}{K_{t+1}}\right)+\left(\frac{1}{\alpha}\right)F_K\left(L_t,K_t;\tau_t\right)\\
&\quad+\left(\frac{1}{2}\right)\left(\frac{I_t}{K_t}\right)^2+\frac{\beta\left(1-\delta\right)}{\alpha}p_{t+1}^I\\
&\quad-\left(\frac{1}{\alpha}\right)p_t^I+\widetilde{\epsilon}_{t+1},
\end{aligned}
$$

where $cons$ is a constant and $\widetilde{\epsilon}_{t+1}$ combines the forecast errors for investment and the purchase price of capital (cf. eq. 20 in Chirinko).

- You see how we have now expressed the benchmark model as a **dynamic investment equation**, where the key ingredients are readily observable (you need to add a parametric expression for F_K - any suggestions?).

- Note: Because the error term is correlated with the regressors (e.g. because forecast errors are correlated with variables in period $t+1$) instrumental variables are needed in estimation.

Potential problems

- Whilst theoretically elegant the Euler equation has not worked very well in practice.

- Notice that the theory implies strong restrictions on what you should get on the right-hand side variables when estimating the Euler equation:

$$\begin{aligned}\left(\frac{I_t}{K_t}\right) &= cons + \beta(1-\delta)\left(\frac{I_{t+1}}{K_{t+1}}\right) + \left(\frac{1}{\alpha}\right) F_K\left(L_t, K_t; \tau_t\right) \\ &+ \left(\frac{1}{2}\right)\left(\frac{I_t}{K_t}\right)^2 + \frac{\beta(1-\delta)}{\alpha} p_{t+1}^I \\ &- \left(\frac{1}{\alpha}\right) p_t^I + \widetilde{\epsilon}_{t+1},\end{aligned}$$

  Very often, what you get in practice, is inconsistent with the theoretical model based on which the Euler equation is derived (e.g. the estimate of $\beta(1-\delta)$ is often larger than 1). This, I think, has been pretty devastating for the Euler equation approach which is now less common in the literature than 10-15 years ago. See Toni Whited's paper entitled "Why do Euler equations fail?" for some clues as to why the investment Euler equation rarely performs well in practice.

- Also, as mentioned above, we need instruments to identify the equation, and finding valid instruments is no easy task.

3 Empirical Research on Investment

There is a large empirical literature investigating the determinants of investment. Lots of different topics and mechanisms have been examined. I will focus on financial constraints and uncertainty. In this section I focus on empirical research in the traditional vein, i.e. research based on linear models suitable for regression analysis. In the final section I provide an introduction to the "new" investment literature, which typically uses a structural approach and goes beyond regression analysis when estimating parameters of interest.

3.1 Application: Investment and Financial Constraints

- Recall that one of the assumptions needed for it to be valid to replace marginal by average q is that investment decisions are made separately from financial decisions.

- This may be a reasonable assumption if financial markets are so well develop so as to make internal and external (debt, new equity) finance perfect substitutes.

- However, in a world where there are "imperfections" in financial markets, the cost of using external funds may exceed the cost of using internal funds.

- To illustrate, forget for a moment about investment dynamics; assume that optimal capital is chosen by the firm so as to equate the marginal revenue product of capital to the marginal cost:

$$MPK = MC.$$

Suppose that MPK is decreasing in capital due to diminishing returns; and suppose using external funds is more expensive than internal funds.

- [Discuss Figure 1 and 2]

Figure 1 Static Demand for Capital

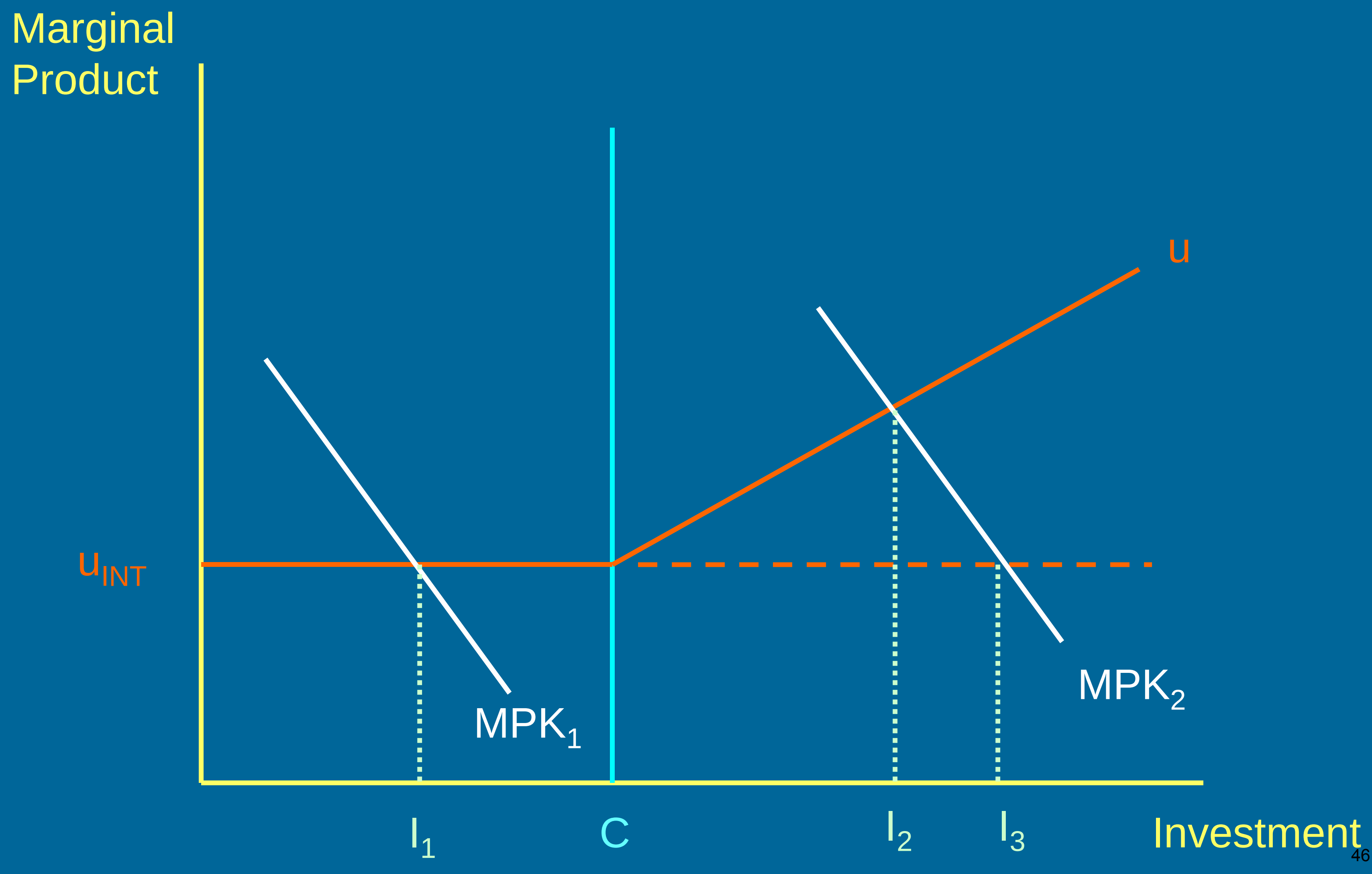
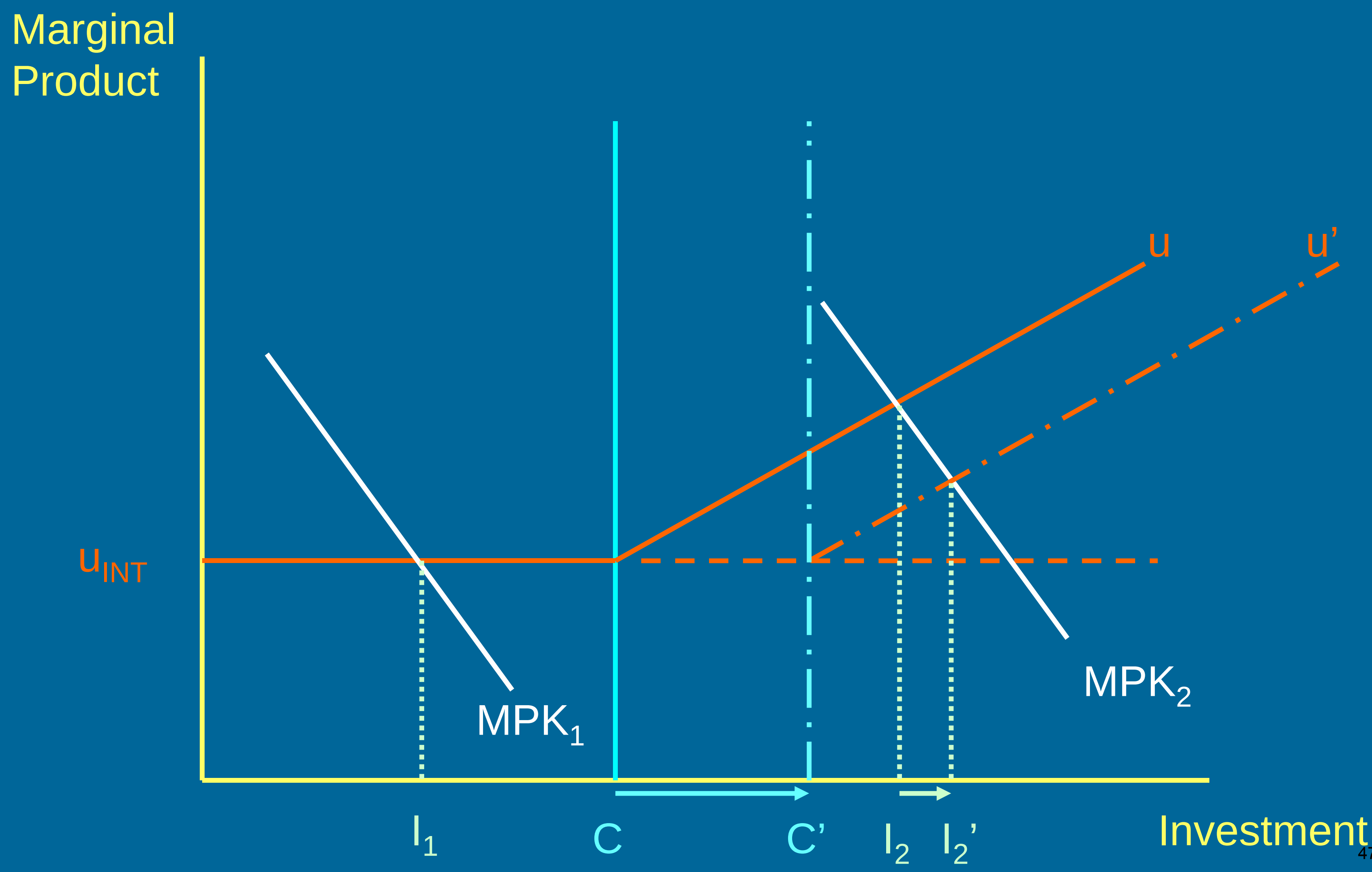


Figure 2 A Cash Flow Shock



- The most common empirical test for financing constraints adopted in the literature is that proposed by Fazzari, Hubbard and Petersen (1988; henceforth FHP). This approach involves investigating the sensitivity of investment to changes in **cash-flow**, conditional on average Q. Average Q, defined as the ratio of the value of the firm to the value of the capital stock, is included in the model in order to take into account other factors than financial constraints that might be affecting investment, for example strong demand or low interest rates.

- The basic idea underlying this 'excess sensitivity' approach is that, under the null hypothesis of no financing imperfections (and a number of other assumptions; see Hayashi, 1982, for details) the only determinant of investment is average Q:

$$\frac{I}{K} = \alpha + \beta \times Q + \varepsilon$$

- If we 'generalize' this equation as follows:

$$\frac{I}{K} = \alpha + \beta Q + \gamma \frac{C}{K} + \varepsilon,$$

where $\frac{C}{K}$ denotes cash-flow divided by the capital stock, we thus have $\gamma = 0$ under the null of no financing constraints (and all other assumptions underlying the q model).

- If cash-flow is found a significant determinant of investment conditional on Q, we say there is **excess sensitivity of investment to cash-flow**.

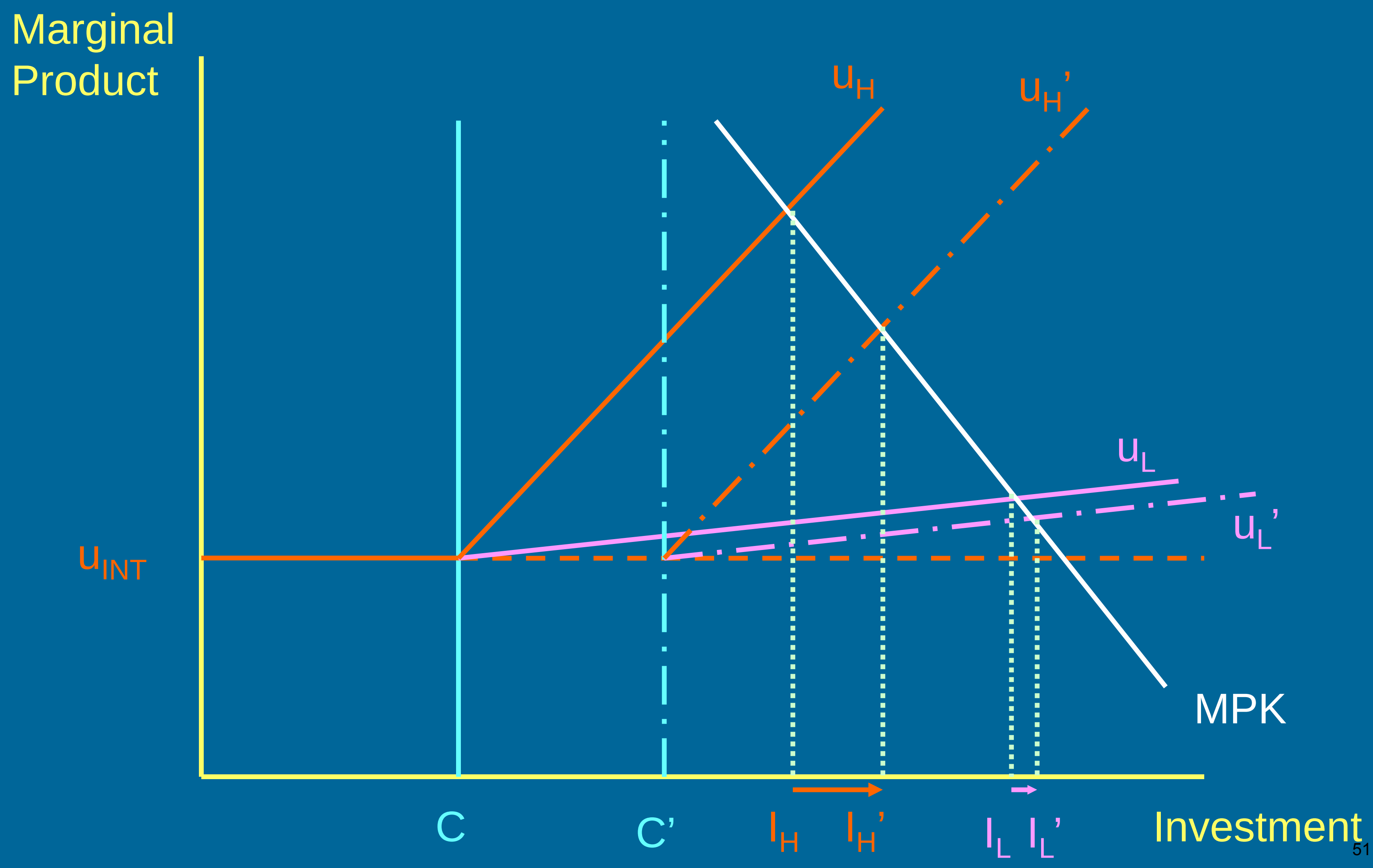
- This means that the null hypothesis that average Q is a sufficient statistic for investment is rejected, which is often taken as a sign of financial imperfections.

- FHP also report results split tests, dividing the sample into a priori 'unconstrained' and 'constrained' sub-samples (e.g. based on size, dividends, credit rating, etc.). They find that
  - the coefficient on cash flow is positive for all sub-samples,
  - that the coefficient on cash flow is larger for 'constrained' sub-samples than for 'unconstrained' subsamples.

- One interpretation (e.g. FHP (1988)): Sub-samples with higher coefficients on cash flow are 'more constrained', e.g. face a higher cost premium for external finance.

[Illustration, Figure 3]

Figure 3 Cost Premia $u_H > u_L$



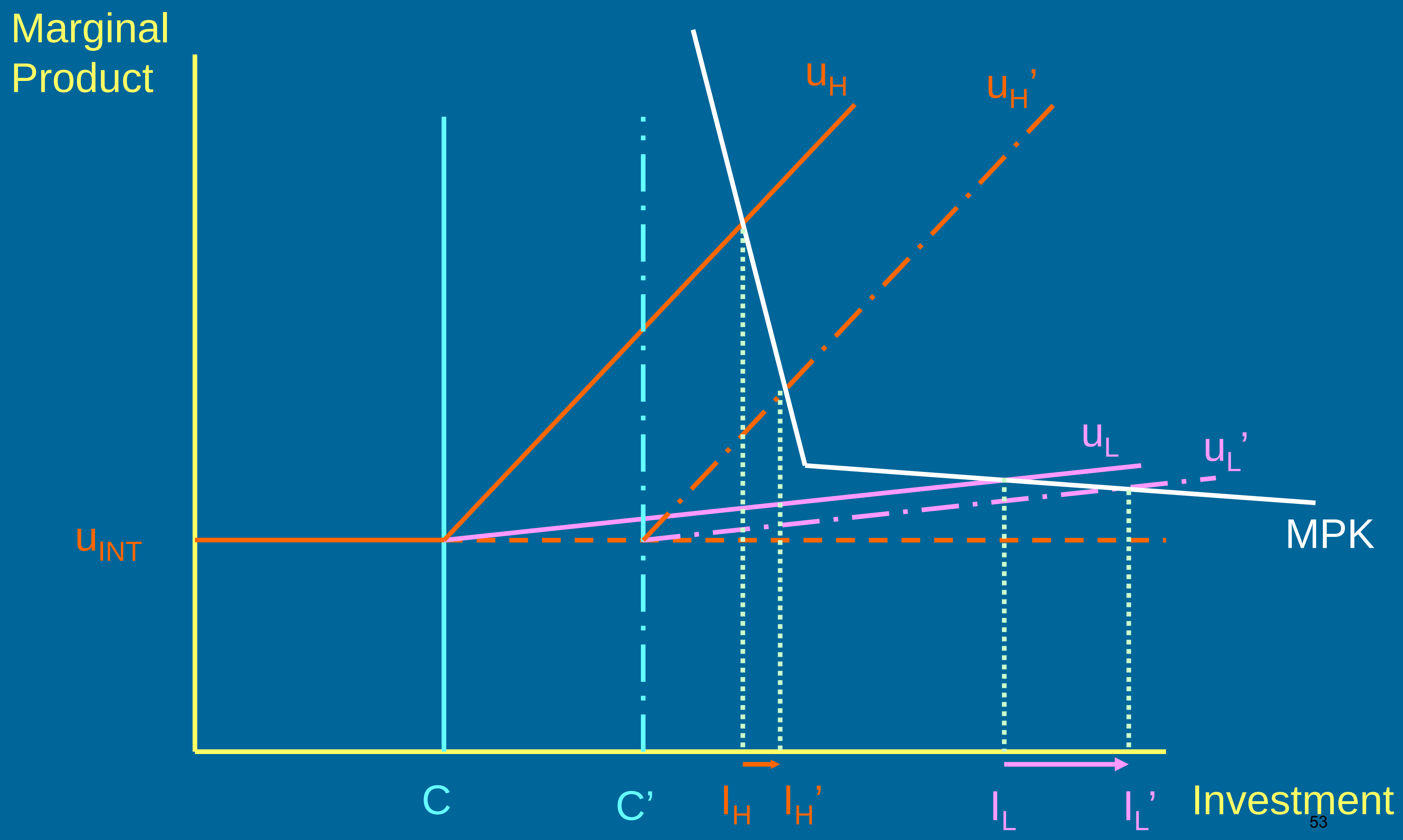
3.1.1 The Kaplan and Zingales (1997) critique

- The approach proposed by Fazzari, Hubbard and Petersen (1988) was very influential in the early and mid 1990s. However, Kaplan and Zingales (1997) argue that this approach is flawed, as cash-flow sensitivities provide no useful information about the severity of financing constraints.

- Kaplan and Zingales show that the investment-cash flow sensitivity may actually be **higher** for firms facing more modest financial constraints, if the marginal product of capital is sufficiently **convex**.

[Illustration, Figure 4]

Figure 4 The Kaplan-Zingales Case



- The Kaplan-Zingales argument is developed for a static model with no adjustment costs, and in which new equity is the only source of external finance, with an increasing cost premium

- Cost premium for external funds implies investment may display excess sensitivity to windfall fluctuations in internal funds.

- But investment-cash flow sensitivity may be **lower** for firms with **higher** cost of external finance, if MPK is sufficiently convex

- This is their key result: **No monotonic relationship** between investment-cash flow sensitivity and the severity of the capital market imperfection

- Numerous authors have accepted this criticism, and consequently eschew the excess sensitivity approach (see e.g. Cleary, 1999; Moyen, 2004).

3.1.2 Bond & Söderbom (2011) on the Kaplan-Zingales critique

- As already noted, the empirical literature on investment and financing constraints building on FHP is typically based on specifications like

$$\frac{I}{K} = \alpha + \beta \times Q + \gamma \left(\frac{C}{K} \right) + \varepsilon.$$

- Formally: a test of the null of no sensitivity to cash flow, conditional on a measure of q, consistent with null of no financing constraints (and otherwise correct specification of the q model, and appropriate measure of q)

- The Kaplan and Zingales (1997) result does **not** invalidate this test, since q is not conditioned on in their model.

- Kaplan and Zingales purport to say something about the value of γ under the alternative. Their own empirical work adopts this specification.

- But their analysis of unconditional investment-cash flow sensitivity in a static demand for capital model may not be informative about conditional investment-cash flow sensitivity in a dynamic investment model with adjustment costs

- Bond-Söderbom emphasize the importance of conditioning on measures of q in order to understand the behaviour of the coefficient on cash flow in such regressions.

- As we have seen, the null specification recognizes role of adjustment costs: capital stock does not adjust to maintain MPK = u, even in absence of financing constraints.

- As we have seen, the relevant FOC equates marginal cost of additional unit of investment with shadow value of additional unit of capital (marginal q).

- The curvature of MPK plays no direct role.

- Usual linear econometric specification further requires marginal adjustment costs to be linear in the investment rate (quadratic adjustment costs).

Questions

- Can we say something about the sensitivity of investment to cash flow conditional on marginal q in an adjustment costs framework?

- Is there a monotonic relationship with the cost premium for external finance?

- Can we measure marginal q using average q in a model with costly external finance?

Approach

- Recall the specification proposed by FHP (and criticized by KZ):

$$\frac{I}{K} = \alpha + \beta \times Q + \gamma \left(\frac{C}{K} \right) + \varepsilon.$$

- This is a theoretically correct specification only if $\gamma = 0$. The model is then consistent with absence of financial imperfections.

- If there are financial imperfections, we know the basic q model is mis-specified. It would seem *plausible* that investment should be sensitive to cash-flow changes in such a case - but we're not sure what the correct model specification would look like.

- Bond-Söderbom generalize the q model to explicitly allow for high external finance costs. All other assumptions needed to replace marginal by average q are maintained (e.g. constant returns, perfect competition etc.) Specifically, they assume the **cost of issuing new equity** (needed to fund investment) is rising at a quadratic rate in the amount issues:

$$\Phi(K_t, N_t) = \left(\frac{\phi}{2}\right)\left(\frac{N_t}{K_t}\right)^2 K_t,$$

where N_t is the amount of new equity and ϕ is a parameter that specifies the slope of the cost premium for external finance. Think of $\Phi(K_t, N_t)$ as a transaction fee that must be paid to third parties when new shares are issued.

- Using debt is also costly, which is modeled by means of an **increasing interest rate schedule**

$$i(K_{t+1}, B_t) = i + \eta \left(\frac{B_t}{K_{t+1}} \right)$$

where i is the interest rate at zero borrowing, B_t is the debt inherited from the last period, and $\eta \geqslant 0$ is a parameter which allows the interest rate to increase with the debt-assets ratio.

- For this generalized version of the model, Bond-Söderbom show that the correct specification of the investment equation is:

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b}\right) + \frac{1}{b}Q_t - \frac{\phi}{b}\left[\left(Q_t - \frac{B_t}{K_{t+1}}\right)\left(\frac{N_t}{K_t}\right)\right]$$

where Q_t denotes average q.

- Intuition?

- Note that this model can be estimated directly, given data on investment rates, average q, debt, and the value of new shares issued. The coefficients estimated are structural parameters of the adjustment cost function or the cost premium function for new equity. Notice that the debt cost premium parameters are not identified from this specification.

- However, we seem to have lost track of the questions with which we began:

  1. Can we say something about the sensitivity of investment to cash flow conditional on marginal q in an adjustment costs framework?

  2. Is there a monotonic relationship with the cost premium for external finance?

  3. Can we measure marginal q using average q in a model with costly external finance?

- In fact, it is straightforward to show that the answer to question (3) is "yes" - see our paper for details, if you are interested.

- To answer (1) and (2), we need to understand how (C/K) correlates with the term $\left(Q_t - \frac{B_t}{K_{t+1}}\right)\left(\frac{N_t}{K_t}\right)$. This is hard (impossible?) to establish analytically, so we use simulations.

- Simulations: Use the correct theoretical model and simulate an artificial panel dataset of firms. Regress investment on average q and cash-flow in the same way as you would with real data.

- [Results in Table 3; Table 4 shows estimates of structural model]

Table 3. Excess Sensitivity Tests: Costly New Equity & Costly Debt

	(i) $\phi = 0$ $\eta = 0$	(ii) $\phi = 1$ $\eta = 0.25$	(iii) $\phi = 2$ $\eta = 1.0$	(iv) $\phi = 4$ $\eta = 20$
Q_t	0.2034	0.1962	0.1894	0.1711
	(.0029)	(.0029)	(.0030)	(.0031)
$\frac{C_t}{K_t}$	-0.0046	0.0129	0.0462	0.1026
	(.0069)	(.0072)	(.0073)	(.0073)
R^2	0.26	0.25	0.26	0.25

See Table 1 for notes.

Table 4. Structural Model Estimates

	(i) $\phi = 0$ $\eta = 0$	(ii) $\phi = 1$ $\eta = 0.25$	(iii) $\phi = 2$ $\eta = 1.0$	(iv) $\phi = 4$ $\eta = 20$
Q_t	0.2021	0.1997	0.2026	0.2012
	(.0021)	(.0021)	(.0021)	(.0021)
$\left(Q_t - \frac{B_t}{K_{t+1}}\right) \times \frac{N_t}{K_t}$	-0.0001	-0.1598	-0.4460	-0.8188
	(.0018)	(.0491)	(.0476)	(.0450)
R^2	0.26	0.25	0.26	0.26

See Table 1 for notes.

Conclusions

- In a benchmark specification with quadratic adjustment costs, increasing costs of external finance, and linear homogeneous functional forms, we find a **monotonic relationship** between this conditional investment-cash flow sensitivity and the cost premia for both new equity and debt finance.

- The Holy Grail in this literature has been an estimable structural model under the imperfect capital markets alternative. We derive a structural investment equation from the first order conditions of our benchmark model

- There are several good reasons why regressions of investment rates on average q and cash flow may not provide reliable evidence about capital market imperfections.

- Even with perfect capital markets:
  - marginal q may not be a sufficient statistic for
  - investment with non-quadratic adjustment costs (more on this below)
  - average q may be a poor proxy for marginal q, due to market power
  - average q may be poorly measured using stock market valuations, due to share price bubbles

- However the non-monotonic relationship between unconditional investment-cash flow sensitivity and the cost premium for external finance, highlighted by Kaplan and Zingales (1997), has little relevance for evaluating this line of research

3.2 Uncertainty and Investment

3.2.1 Leahy and Whited (1996)

John Leahy and Toni Whited (1996): "The Effects of Uncertainty on Investment: Some stylized facts". Journal of Money, Credit and Banking.

Brief overview

- Panel estimation of the effect of uncertainty on investment
- Approach: Use yearly **volatility** of daily returns of stock as a measure of uncertainty.

- Estimate yearly firm investment (from COMPUSTAT database) as a function of this

- Use firm and year controls to try and deal with other omitted variables

- Use GMM to try to deal with endogeneity

- Key result: Uncertainty has a negative influence on investment.

- This mechanism appears to operate through Tobin's q (high uncertainty $\rightarrow$ low q).

- **Note:** No strong link between theory and empirics.

Leahy and Whited (1996)

Basic specification (see paper for definitions):

$$\Delta\left(\frac{I}{K}\right)_{it} = \Delta c_t + \sum_{n=0}^{N} \alpha_n \Delta E_t \sigma_{i,t+n} + \Delta u_{it} . \qquad (2)$$

Table 2: Effect of one-period uncertainty forecasts on investment

Tobin's q	0.024 (0.009)	—	0.022 (0.023)
Variance	—	−0.538 (0.276)	−0.054 (0.536)

The sample consists of 600 U.S. manufacturing firms observed 1982-1987.
The dependent variable is Investment / Capital Stock. Standard errors in ( ).

3.2.2 Guiso and Parigi (1999)

Luigi Guiso and Giuseppe Parigi (1999). "Investment and Demand Uncertainty". Quarterly Journal of Economics.

Brief overview

- Estimates the effect of uncertainty on investment

- Measure of uncertainty is based on a survey of Italian firms' **subjective probability distribution** of demand growth expectations.

- Essentially firms were asked to indicate the perceived probability that demand would: a) grow by more than 50%; b) grow 25-50%; c) grow 15-25%; and so on, until; shrink by more than 15%.

- The authors then use these survey data to generate a mean and variance of expected demand.

- Basic specification:

$$\frac{{}_0I_1^p}{K_0} = \alpha_0 + \alpha_1 \frac{{}_0y_i}{K_0}\left(1 - \alpha_{20}u_i\right) + \alpha_3 \frac{I_0}{K_{-1}} + \alpha_4 Z_i + \epsilon_1,$$

where $\frac{{}_0I_1^p}{K_0}$ is investment planned by the firm at the end of year 0 for year 1; I_0 is the investment made in year 0; K denotes capital, ${}_0y_i$ is the level of demand expected at the end of year 0 for year i, ${}_0u_i$ is the measure of subjective uncertainty, Z_i is a vector of control variables and ϵ_1 is an error term.

- Note that they estimate effects of variance controlling for the mean.

- Note also they are primarily interested in the interaction term uncertainty x expected demand. The idea is that, if investments are **irreversible**, the effect of an increase in uncertainty is to reduce the responsiveness of investment to demand shocks. (Level of uncertainty is included in the Z_i vector, as a control).

- Main message: uncertainty reduces responsiveness.

- Again: No strong link between theory and empirics.

[Table III here]

TABLE III

INVESTMENT AND UNCERTAINTY

DEPENDENT VARIABLE: RATIO OF INVESTMENT PLANNED ONE YEAR AHEAD TO THE STOCK OF CAPITAL

Variable	Expectations three years ahead (1)	(2)	(3)	(4)	Expectations one year ahead (5)	(6)
$\frac{{}_0y_i}{K_0}$	0.0069	0.0072	0.0054	0.0064	0.0085	0.0072
	(0.0011)	(0.0011)	(0.0005)	(0.0011)	(0.0012)	(0.0012)
$\left(\frac{{}_0y_i}{K_0}\right)\left(\frac{{}_0\sigma_i}{K_0}\right)$	−0.0081	−0.0089	—	−0.0079	−0.0084	−0.0069
	(0.0015)	(0.0016)		(0.0015)	(0.0020)	(0.0019)
$\frac{I_0}{K_1}$	0.4249	0.4361	0.4302	0.3956	0.4594	0.4431
	(0.0185)	(0.0186)	(0.0189)	(0.0189)	(0.0177)	(0.0179)
$\frac{({}_0y_i)^2}{K_0}$	—	−1.25E-09	—	—		
		(1.01E-09)				
$\frac{{}_0\sigma_i}{K_0}$	—	0.0281	—	—		
		(0.0212)				
$\left(\frac{{}_0y_i}{K_0}\right){}_0\sigma^2_{di}$	—	—	−0.1974			
			(0.0814)			
$\frac{CF}{K_0}$	—	—	—	0.0078	—	0.0139
				(0.0048)		(0.0053)
RAT	—	—	—	−0.0028	—	−0.0020
				(0.0041)		(0.0038)
Constant	0.0041	0.0026	0.0067	0.0072	0.0030	0.0044
	(0.0122)	(0.0121)	(0.0122)	(0.0121)	(0.0114)	(0.0113)
Number of observations	549	549	549	514	603	568
F-test for all coefficients = 0	26.63	26.10	29.20	20.57	32.02	26.27
	(29, 519)	(34, 514)	(32, 516)	(34, 479)	(32, 570)	(34, 533)

Source: Guiso and Parigi, QJE, 1999.

4 Recent Developments in the Literature

Reference: Bloom, Nicholas (2009), "The Impact of Uncertainty Shocks," Econometrica 77, 623-685. (Bloom was awarded the Frisch Medal of the Econometric Society for this paper.)

- The primary contribution of this paper is to analyze the effects of uncertainty **shocks** on various important micro and macro quantities using a structural approach.

- Notice the emphasis on shocks: we are interested in the effects of **changes** in the second moment.

- That there are such shocks to uncertainty seems hard to dispute - e.g. the 9/11 attacks. Uncertainty, of course, is hard to measure. However, using data from financial markets we can learn quite a bit about the market's sentiment of risk.

- More specifically, we can back out **implied volatility** on a particular share by using data on the price of the associated **option** combined with data on its theoretical determinants (e.g. stock price, exercise price of option, interest rate etc.).

- To illustrate, consider the Black-Scholes formula for the theoretical price C of a European call option, giving the holder the right to buy one share at price K after T years:

$$C = S\Phi\left(d_1\left(\sigma\right)\right) - Ke^{-rT}\Phi\left(d_2\left(\sigma\right)\right),$$

where S is the current price of the stock, $\Phi(.)$ is the cumulative density function for the standard normal distribution, r is the risk-free interest rate, and

$$d_1 = \frac{\ln(S/K) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}},$$

and

$$d_2 = d_1 - \sigma\sqrt{T},$$

where σ is the standard deviation of returns. Clearly if you know all the ingredients of this formula except σ, you can back out σ rather easily.

- The Chicago Board Options Exchange (CBOE) publishes an index known as the Risk Sentiment Indicator, or the VXO index, which is based on the trading of S&P 100 (OEX) options.

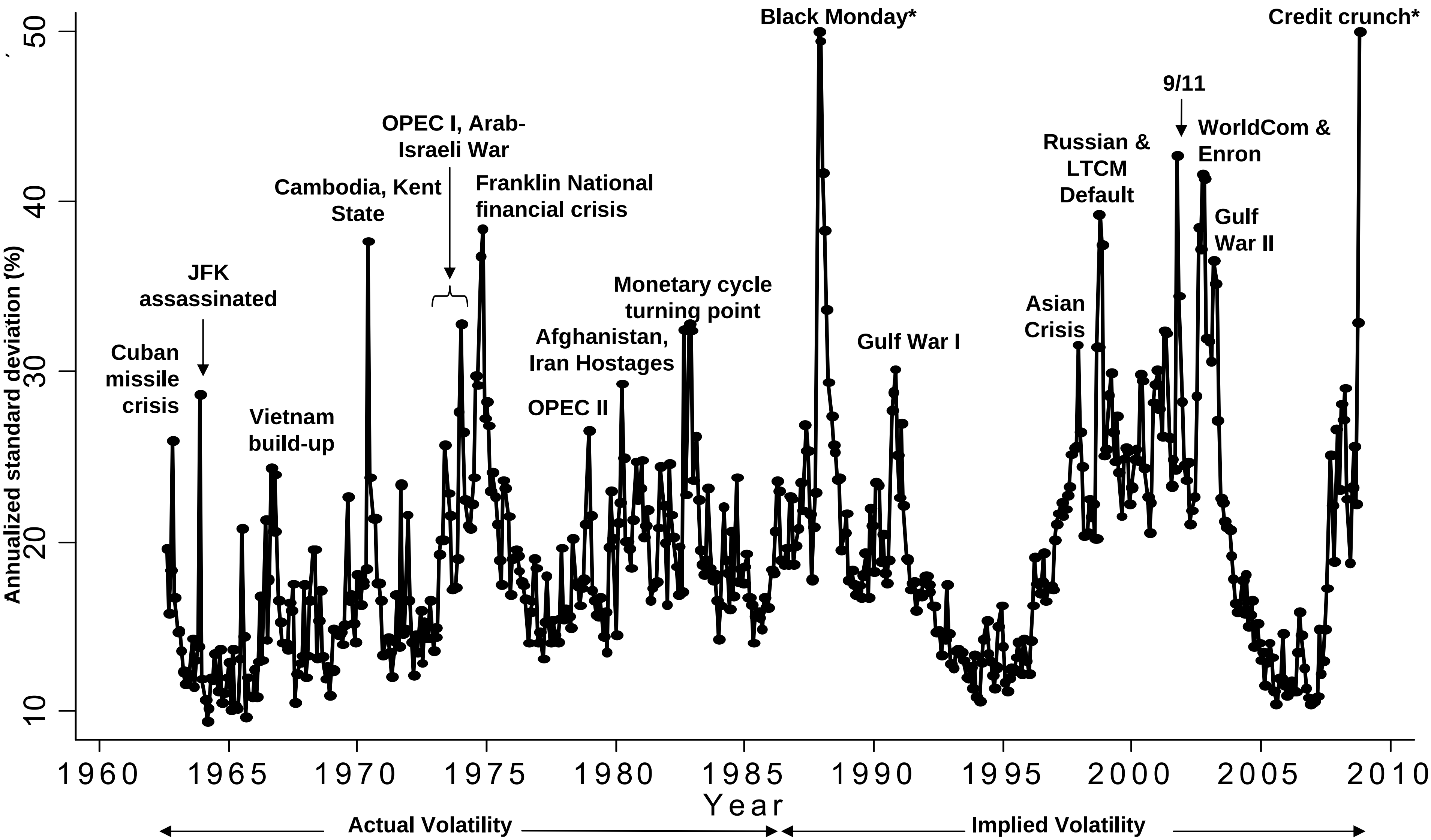
- This index is interpretable as the **annualized standard deviation in returns**. Data on the VXO index are available from 1986. Figure 1 in Bloom shows a time series plot of the VXO index, combined with monthly standard deviation of the daily S&P500 index for the period before 1986. The graph shows two important facts:

- There is a lot of variation over time in perceived variability of returns. Volatility doubles at times of major shocks.

- Perceived variability of stock market returns tends to be high at times of major economic and political shocks. If you look carefully in the notes, you see that the index reached a 45-year high at the recent credit crunch peak.

[Figure 1 here]

Figure 1: Monthly US stock market volatility



Notes: CBOE VXO index of % implied volatility, on a hypothetical at the money S&P100 option 30 days to expiration, from 1986 onwards. Pre 1986 the VXO index is unavailable, so actual monthly returns volatilities calculated as the monthly standard-deviation of the daily S&P500 index normalized to the same mean and variance as the VXO index when they overlap from 1986 onwards. Actual and VXO are correlated at 0.874 over this period. The market was closed for 4 days after 9/11, with implied volatility levels for these 4 days interpolated using the European VX1 index, generating an average volatility of 58.2 for 9/11 until 9/14 inclusive. A brief description of the nature and exact timing of every shock is contained in Appendix A. Shocks defined as events 1.65 standard deviations about the Hodrick-Prescott detrended (λ=129,600) mean, with 1.65 chosen as the 5% significance level for a one-tailed test treating each month as an independent observation.

* For scaling purposes the monthly VXO was capped at 50. Un-capped values for the Black Monday peak are 58.2 and for the Credit Crunch peak are 64.4

- This paper adopts a structural approach. Writes down a theoretical investment model, and uses real data to estimate the model parameters. Then analyzes effects of uncertainty shocks.

- Highlights of the model predictions.

  - An uncertainty shock yields a rapid slowdown (and bounceback) in investment.

  - Right after an uncertainty shock firms are unresponsive to price changes. Potentially important from a policy point of view - in such a situation policy may be pretty ineffective

  - More on policy: trade-off between policy "correctness" and policy "decisiveness" - it may be it may be better to act decisively (but occasionally incorrectly) then to deliberate on policy, generating policy-induced uncertainty.

4.1 The model

- Bloom's model is extension of the standard model of the firm reviewed above (Chirinko's class of 'explicit models'), in two ways:
  - Uncertainty is modelled as a **stochastic process**, i.e. the variance parameter is affected by shocks and is therefore not constant
  - There is a **mix** of convex and non-convex adjustment costs, affecting hiring and investment decisions. The non-convex adjustment costs are crucial, generating real option effects.

4.1.1 The revenue function

- Cobb-Douglas production function exhibiting constant returns to scale:

$$F = \tilde{A} K^{\alpha} (LH)^{1-\alpha} ,$$

where $\tilde{A}$ denotes productivity, K is capital, L is labour, and H is hours worked.

- Iso-elastic demand for the firm's product:

$$Q = B \times P^{-\epsilon},$$

where B is a stochastic demand shifter and $-\epsilon < -1$ is the price elasticity of demand (i.e. if $-\epsilon$ is a large negative, then the price elasticity is high & the demand curve fairly flat).

- Combining the production function and the demand equation assuming

$F = Q$, we get the revenue function:

$$\begin{aligned} R &= P \times F \\ R &= (F/B)^{-\frac{1}{\epsilon}} \times F \\ R &= B^{\frac{1}{\epsilon}} \times F^{\frac{\epsilon-1}{\epsilon}} \\ R &= B^{\frac{1}{\epsilon}} \left(\tilde{A} K^{\alpha} (LH)^{1-\alpha}\right)^{\frac{\epsilon-1}{\epsilon}}. \end{aligned}$$

For notation clarity, write this as

$$S = A^{1-a-b} K^{a} (LH)^{b},$$

where

$$\begin{aligned} A^{1-a-b} &= B^{\frac{1}{\epsilon}} \tilde{A}^{\frac{\epsilon-1}{\epsilon}}, \\ a &= \alpha \left(\frac{\epsilon - 1}{\epsilon}\right), \\ b &= (1 - \alpha) \left(\frac{\epsilon - 1}{\epsilon}\right) \end{aligned}$$

(you should confirm this). Notice that the revenue function S is homogeneous of degree 1 in $A, K, (LH)$, which, as we shall see later, is a very useful property. From now on, refer to A as the 'business conditions' parameter.

- Wages are specified as

$$w(H) = w_1 (1 + w_2 H^\gamma),$$

where w_1, w_2, γ are parameters to be estimated.

- Capital depreciates at a fixed rate δ_K, and there is an exogenous labour quit rate of δ_L.

4.1.2 The stochastic process for demand & productivity

- Business conditions A are modelled as an augmented **geometric random walk:**

$$A_{i,j,t} = A_t^M \times A_{i,t}^F \times A_{i,j,t}^U,$$

where A_t^M is a macro-level component; $A_{i,t}^F$ is a firm-level component; $A_{i,j,t}^U$ is a unit-level (e.g. plant) component; and i, j, t index firm, unit (plant) and time, respectively.

- The macro component.

$$A_t^M = A_{t-1}^M \left(1 + \sigma_{t-1} W_t^M\right),$$

where σ_{t-1} is the standard deviation of business conditions and W_t^M is a macro-level i.i.d. shock drawn from a standard normal distribution,. $W_t^M \sim N(0, 1)$.

- The firm-level component:

$$A_{i,t}^{F} = A_{i,t-1}^{F}\left(1 + \mu_{i,t} + \sigma_{t-1}W_{i,t}^{F}\right),$$

where $\mu_{i,t}$ is a firm-level **drift** in business conditions, $W_{i,t}^{F}$ is a firm-level i.i.d. shock drawn from a standard normal distribution,. $W_{i,t}^{F}\tilde{}N(0,1)$.

- The unit-level component:

$$A_{i,j,t}^{U} = A_{i,j,t-1}^{U}\left(1 + \sigma_{t-1}W_{i,j,t}^{U}\right),$$

where $W_{i,,j,t}^{U}$ is a firm-level i.i.d. shock drawn from a standard normal distribution,. $W_{i,j,t}^{U}\tilde{}N(0,1)$.

- The shocks $W_{t}^{M}, W_{i,t}^{F}, W_{i,j,t}^{U}$ are all assumed independent of each other. Notice also that the uncertainty parameter is the **same** across the previous three specifications - i.e. macro, firm and unit uncertainty are the same!

- The stochastic volatility (uncertainty) process $\left(\sigma_t^2\right)$ and the demand conditions drift $\left(\mu_{i,t}\right)$ are assumed to follow **two-point Markov chains:**

$$\begin{aligned} \sigma_t &\in \{\sigma_L, \sigma_H\} \qquad \text{where } \Pr\left(\sigma_{t+1} = \sigma_j | \sigma_t = \sigma_k\right) = \pi_{k,j}^{\sigma} \\ \mu_{i,t} &\in \{\mu_L, \mu_H\} \qquad \text{where } \Pr\left(\mu_{t+1} = \mu_j | \mu_t = \mu_k\right) = \pi_{k,j}^{\mu}. \end{aligned}$$

  That is, these variables take one of two values, and the transition probabilities are given by $\pi_{k,j}^{\sigma}$ and $\pi_{k,j}^{\mu}$.

4.1.3 Adjustment costs

Three terms:

1. **Partial irreversibilities**.

- Cost of hiring and firing workers:

$$C_L^P \times 52w\,(40)\left[E^+ + E^-\right],$$

where C_L^P is a fraction of annual wages and E^+,E^- denote absolute hiring and firing

- Cost of purchasing and selling off capital (the latter due to transaction costs, market for lemons etc.):

$$\left[I^+ - \left(1 - C_K^P\right) \times I^-\right],$$

where C_K^P is the resale loss of capital denominated as a fraction of the relative purchase price of capital, and I^+,I^- denote absolute values of investment and disinvestment.

- C_L^P and C_K^P are parameters to be estimated. High values imply high costs and high real option values - encouraging wait-and-see decisions.

Over a range of values for the business condition parameter, the firm chooses to do nothing - zero hiring and firing, zero investment. This is known as the region of inaction.

2. **Fixed disruption costs**. When the level of employment or the level of capital stock change, there may be a fixed loss of output. You may have to shut down the factory for a few days when installing new capital, for example. These fixed costs are denoted by C_L^F and C_L^F, for capital and labour, respectively, both denominated as fractions of annual sales:

$$C_L^F \mathbf{1}_{[E \neq 0]} \times S$$
$$C_K^F \mathbf{1}_{[I \neq 0]} \times S.$$

If fixed costs are high, it makes sense for the firm to do a lot of adjustment or none at all; i.e. adjustment tends to be "lumpy".

3. **Quadratic adjustment costs:**

$$C_L^Q \times L \left(\frac{E}{L}\right)^2,$$
$$C_K^Q \times K \left(\frac{I}{K}\right)^2$$

We saw above that this was the standard form of adjustment costs in the literature during the 1980s and early 1990s. The idea is that large changes to employment or capital are very costly. If quadratic adjustment costs are high, it makes sense for the firm to spread out a given adjustment over several periods, generating smooth and continuous adjustment towards the long-run target.

Total adjustment costs are thus given by

$$\begin{aligned} C \;=\; & C_L^P \times 52w\,(40)\left[E^+ + E^-\right] + \left[I^+ - \left(1 - C_K^P\right) \times I^-\right] \\ & + C_L^F \mathbf{1}_{[E\neq 0]} \times S + C_K^F \mathbf{1}_{[I\neq 0]} \times S \\ & + C_L^Q \times L\left(\frac{E}{L}\right)^2 + C_K^Q \times K\left(\frac{I}{K}\right)^2 . \end{aligned}$$

4.1.4 Optimal investment and employment

The firm's optimization problem is to maximize the present discounted flow of revenues less the wage bill and the adjustment costs:

$$V\left(A_t, K_t, L_t, \sigma_t, \mu_t\right) = \max_{I_t, E_t, H_t} E_t \left\{ \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \left[S_t - C_t - w_t L_t\right] \right\},$$

where V denotes the value of the firm, r is the one-period (constant) discount rate, $E_t[.]$ denotes an expected value given information available at time t, and

$$\begin{aligned} S_t &= S\left(A_t, K_t, L_t, H_t\right) \quad \text{(revenues)} \\ C_t &= C\left(A_t, K_t, L_t, H_t, I_t, E_t\right) \quad \text{(adjustment costs)} \\ w_t &= w\left(H_t\right) \quad \text{(wage rate).} \end{aligned}$$

Using recursive methods, we can expressed the firm's optimization problem as a Bellman equation:

$$V\left(A_t, K_t, L_t, \sigma_t, \mu_t\right) =$$

$$\max_{I_t, E_t, H_t} \left\{ \begin{array}{c} S_t - C_t - w_t L_t + \\ \left(\frac{1}{1+r}\right) E_t V\left(A_{t+1}, K_{t+1}, L_{t+1}, \sigma_{t+1}, \mu_{t+1}\right) \end{array} \right\}$$

This is equation simplified in two important ways:

1. Since hours (H_t) is a flexible factor it can be optimized out in a prior step, using a conventional static first order condition equalizing the marginal cost of hours to its marginal revenue. Optimal level of hours can be written as a function of predetermined variables and parameters of the model, and so we can replace hours by its determinants in the maximization problem. This means we don't have to solve numerically for hours.

2. Since the value function V is homogeneous of degree 1 in (A_t, K_t, L_t), we can normalize the value function by capital and write:

$$Q\left(a_t, l_t, \sigma_t, \mu_t\right) = \max_{i_t, e_t} \left\{ \begin{array}{c} S^*\left(a_t, l_t\right) - C^*\left(a_t, l_t, i_t, l_t e_t\right) + \\ \left(\frac{1-\delta_K + i_t}{1+r}\right) E_t Q\left(a_{t+1}, l_{t+1}, \sigma_{t+1}, \mu_{t+1}\right) \end{array} \right\},$$

where

$$
\begin{aligned}
Q &= V/K \\
a &= A/K \\
l &= L/K \\
e &= E/L
\end{aligned}
$$

are normalized variables, and $S^*(a_t, l_t)$ and $C^*(a_t, l_t, i_t, l_t e_t)$ are sales and costs (both normalized by K) after optimization over hours. Note that Q is interpretable as Tobin's Q.

4.1.5 Aggregation

- Plant-level data typically indicate that hiring and investment are lumpy with lots of zeros. In firm-level data, however, investment and hiring are much smoother. Bloom has firm-level data, and therefore aggregates unit (plant) level data into firm-level data, assuming that each firm consists of 250 units.

4.1.6 How this model is used

Recap:

- The ultimate goal of the paper is to document the effects of uncertainty shocks on several quantities of interest, e.g. employment, investment and productivity.

- These effects are **inferred (**simulated**)** from the model outlined in the previous section

- The model, of course, contains a lot of unknown parameters, and the effect of uncertainty shocks will depend crucially on the values of those parameters.

- For example, if irreversibilities are important (i.e. C_K^P is high), this will result in firms postponing their investments.

- So before we say anything about these effects, we need to estimate the unknown parameters of the model. Estimation of the model parameters is a difficult task in practice. But the overall principles are straightforward.

1. First, conditional on a given vector of parameter values, we solve for optimal investment and hiring, using the model above. Unfortunately, this is not straightforward and can't be done analytically. Bloom uses numerical dynamic programming. In the appendix, I provide an illustration of one popular numerical dynamic programming technique known as value iteration.

2. Second, based on these solutions we compare the predicted outcomes of the model - investment, hiring, output etc - to real outcomes in data. The aim is to mimic the real data as closely as possible, which is the basis for estimation: we vary the structural parameters until the model predictions are as close as they can be to real outcomes. At this point we have obtained our estimates of the structural parameters.

- Equipped with the estimates of the structural parameters, we can carry out **counterfactual simulations** in order to analyze the effects of uncertainty shocks. We can ask, for example, what happens to investment (according to the model) when uncertainty changes from a low level (σ_L) to a high level (σ_H). This type of analysis is done in Section 4, in Bloom's paper.

4.1.7 Principles of estimation

- Basis for estimation: Can **infer** adjustment costs (and other parameters) from observed **moments** in the real data. For example:
  - If lots of zeros in investment data $=>$ quadratic costs not the whole story

  - If high serial correlation in investment rates => fixed costs not the whole story

  - If lots of large investments in data => fixed costs likely

  - If low correlation between investment and sales growth => high quadratic costs likely

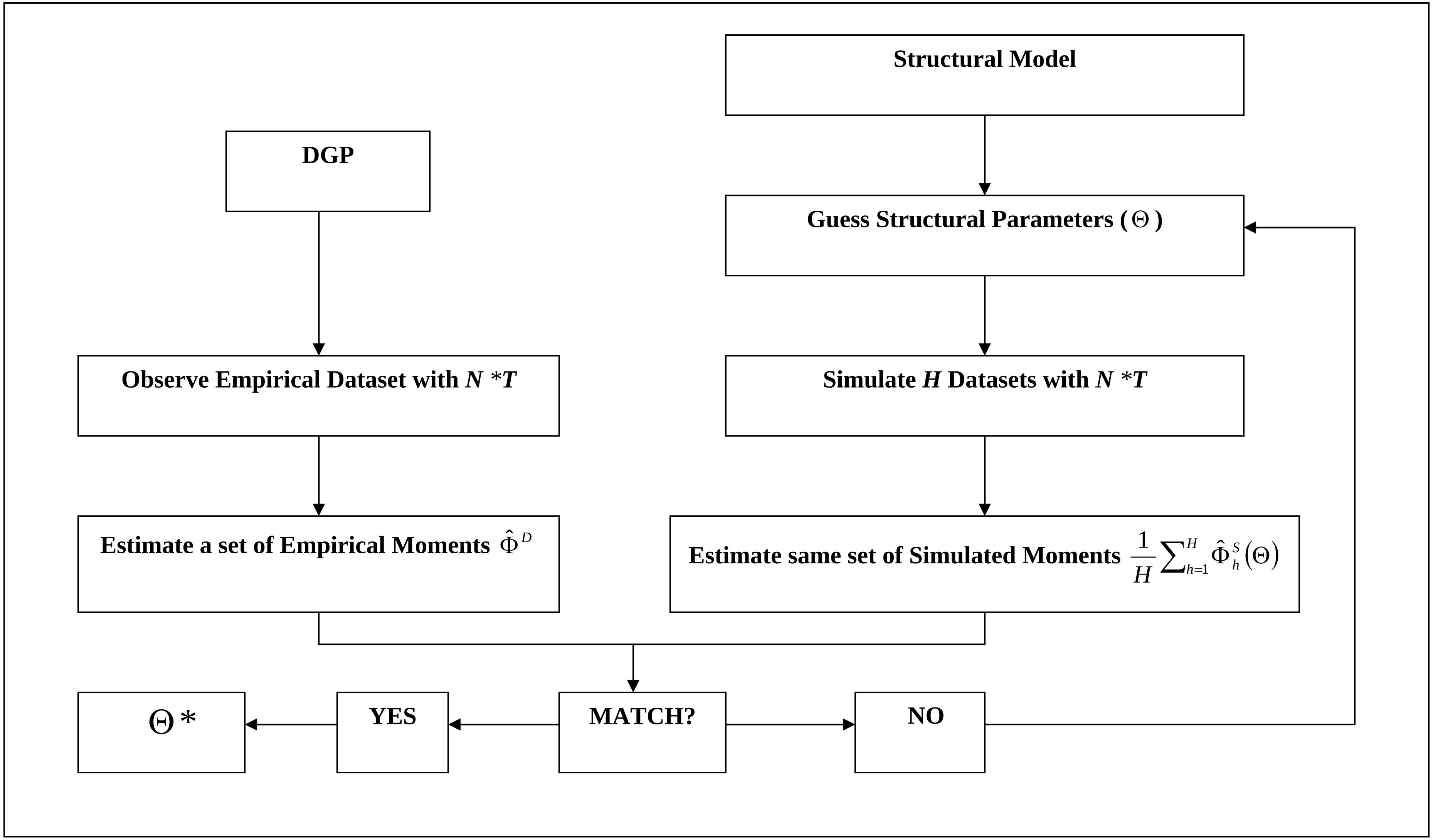
- **Method of simulated moments** (McFadden, 1989). Very flexible and relatively easy to implement.

- The idea is quite intuitive: different parameter values give rise to different observable patterns (moments) in the data.

- Moments **simulated** from structural model. Vary parameter values, with the objective of obtaining the best possible match between simulated & real moments.

[Diagram for SMM here]

[Bloom moments and results here]



Structural Model
DGP
Guess Structural Parameters (Θ)
Observe Empirical Dataset with $N*T$
Simulate H Datasets with $N*T$
Estimate a set of Empirical Moments $\hat{\Phi}^D$
Estimate same set of Simulated Moments $\frac{1}{H}\sum_{h=1}^{H}\hat{\Phi}_h^S(\Theta)$
$\Theta*$
YES
MATCH?
NO

Table 3: Adjustment cost estimates

Adjustment Costs Specification:		All	Capital	Labor	Quad	None
Estimated Parameters:						
$\mathbf{C}_K^P$		33.9	42.7			
investment resale loss (%)		*(6.8)*	*(14.2)*			
$\mathbf{C}_K^F$		1.5	1.1			
investment fixed cost (% annual sales)		*(1.5)*	*(0.2)*			
$\mathbf{C}_K^Q$		0	0.996		4.844	
capital quadratic adjustment cost (parameter)		*(0.009)*	*(0.044)*		*(454.15)*	
$\mathbf{C}_L^P$		1.8		16.7		
per capita hiring/firing cost (% annual wages)		*(0.8)*		*(0.1)*		
$\mathbf{C}_L^F$		2.1		1.1		
fixed hiring/firing costs (% annual sales)		*(0.9)*		*(0.1)*		
$\mathbf{C}_L^Q$		0		1.010	0	
labor quadratic adjustment cost (parameter)		*(0.037)*		*(0.017)*	*(0.002)*	
$\boldsymbol{\sigma}_L$		0.443	0.413	0.216	0.171	0.100
baseline level of uncertainty		*(0.009)*	*(0.012)*	*(0.005)*	*(0.005)*	*(0.005)*
$\boldsymbol{\mu}_H - \boldsymbol{\mu}_L$		0.121	0.122	0.258	0.082	0.158
spread of firm business conditions growth		*(0.002)*	*(0.002)*	*(0.001)*	*(0.001)*	*(0.001)*
$\boldsymbol{\pi}_{H,L}^{\mu}$		0	0	0.016	0	0.011
transition of firm business conditions growth		*(0.001)*	*(0.001)*	*(0.001)*	*(0.001)*	*(0.001)*
$\boldsymbol{\gamma}$		2.093	2.221	3.421	2.000	2.013
curvature of the hours/wages function		*(0.272)*	*(0.146)*	*(0.052)*	*(0.009)*	*(14.71)*
Moments:	*Data*	Data moments - Simulated moments				
Correlation $(I/K)_{i,t}$ with $(I/K)_{i,t-2}$	*0.328*	0.060	-0.015	0.049	-0.043	0.148
Correlation $(I/K)_{i,t}$ with $(I/K)_{i,t-4}$	*0.258*	0.037	0.004	0.088	0.031	0.162
Correlation $(I/K)_{i,t}$ with $(\Delta L/L)_{i,t-2}$	*0.208*	0.003	-0.025	0.004	-0.056	0.078
Correlation $(I/K)_{i,t}$ with $(\Delta L/L)_{i,t-4}$	*0.158*	-0.015	-0.009	0.036	0.008	0.091
Correlation $(I/K)_{i,t}$ with $(\Delta S/S)_{i,t-2}$	*0.260*	-0.023	-0.062	-0.044	-0.102	0.024
Correlation $(I/K)_{i,t}$ with $(\Delta S/S)_{i,t-4}$	*0.201*	-0.010	-0.024	0.018	-0.036	0.087
Standard Deviation $(I/K)_{i,t}$	*0.139*	-0.010	0.010	-0.012	0.038	0.006
Coefficient of Skewness $(I/K)_{i,t}$	*1.789*	0.004	0.092	1.195	1.311	1.916
Correlation $(\Delta L/L)_{i,t}$ with $(I/K)_{i,t-2}$	*0.188*	-0.007	0.052	-0.075	0.055	0.053
Correlation $(\Delta L/L)_{i,t}$ with $(I/K)_{i,t-4}$	*0.133*	-0.021	0.024	-0.061	0.038	0.062
Correlation $(\Delta L/L)_{i,t}$ with $(\Delta L/L)_{i,t-2}$	*0.160*	0.011	0.083	-0.033	0.071	0.068
Correlation $(\Delta L/L)_{i,t}$ with $(\Delta L/L)_{i,t-4}$	*0.108*	-0.013	0.054	-0.026	0.045	0.060
Correlation $(\Delta L/L)_{i,t}$ with $(\Delta S/S)_{i,t-2}$	*0.193*	-0.019	0.063	-0.091	0.064	0.023
Correlation $(\Delta L/L)_{i,t}$ with $(\Delta S/S)_{i,t-4}$	*0.152*	0.003	0.056	-0.051	0.059	0.063
Standard Deviation $(\Delta L/L)_{i,t}$	*0.189*	-0.022	-0.039	0.001	-0.001	0.005
Coefficient of Skewness $(\Delta L/L)_{i,t}$	*0.445*	-0.136	0.294	-0.013	0.395	0.470
Correlation $(\Delta S/S)_{i,t}$ with $(I/K)_{i,t-2}$	*0.203*	-0.016	-0.015	-0.164	-0.063	-0.068
Correlation $(\Delta S/S)_{i,t}$ with $(I/K)_{i,t-4}$	*0.142*	-0.008	-0.010	-0.081	-0.030	-0.027
Correlation $(\Delta S/S)_{i,t}$ with $(\Delta L/L)_{i,t-2}$	*0.161*	-0.005	0.032	-0.105	-0.024	-0.037
Correlation $(\Delta S/S)_{i,t}$ with $(\Delta L/L)_{i,t-4}$	*0.103*	-0.015	0.011	-0.054	-0.005	-0.020
Correlation $(\Delta S/S)_{i,t}$ with $(\Delta S/S)_{i,t-2}$	*0.207*	-0.033	0.002	-0.188	-0.040	-0.158
Correlation $(\Delta S/S)_{i,t}$ with $(\Delta S/S)_{i,t-4}$	*0.156*	0.002	0.032	-0.071	-0.021	-0.027
Standard Deviation $(\Delta S/S)_{i,t}$	*0.165*	0.004	0.003	0.033	0.051	0.062
Coefficient of Skewness $(\Delta S/S)_{i,t}$	*0.342*	-0.407	-0.075	-0.365	0.178	0.370
Criterion, $\Gamma(\theta)$		404	625	3618	2798	6922

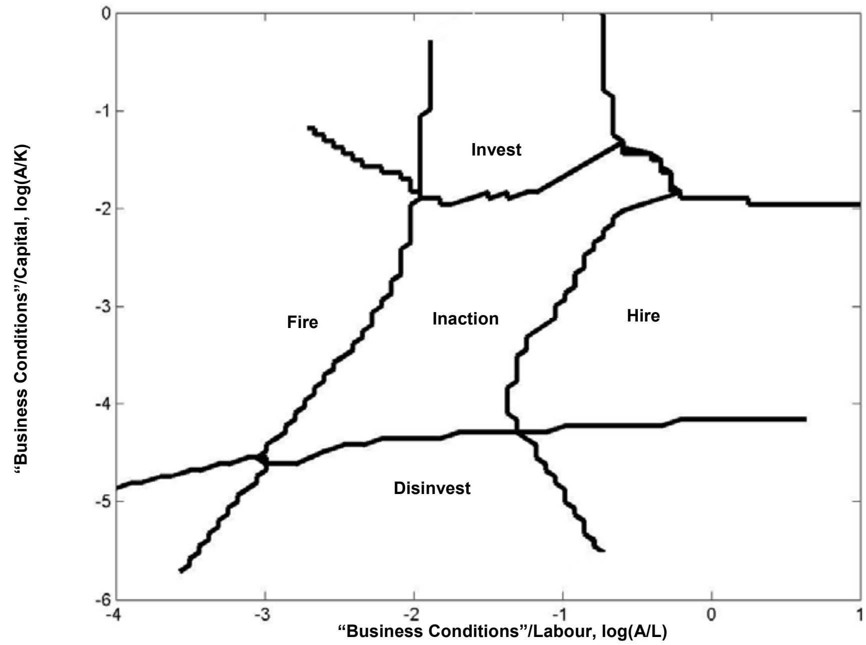
4.1.8 Summary of results and simulations

- Significant region of inaction (figure 5), due to non-convex adjustment costs. Firms only hire and invest when business conditions are sufficiently good. When uncertainty is higher, the region of inaction expands. This suggests that large changes in σ_t can have an important impact on investment and hiring.

- The parameterized model is used to simulate a large macro uncertainty shock, which produces a rapid drop and rebound in output, employment and productivity growth (see e.g. Figure 8). This is due to the effect of higher uncertainty making firms temporarily pause their hiring and investment behavior.

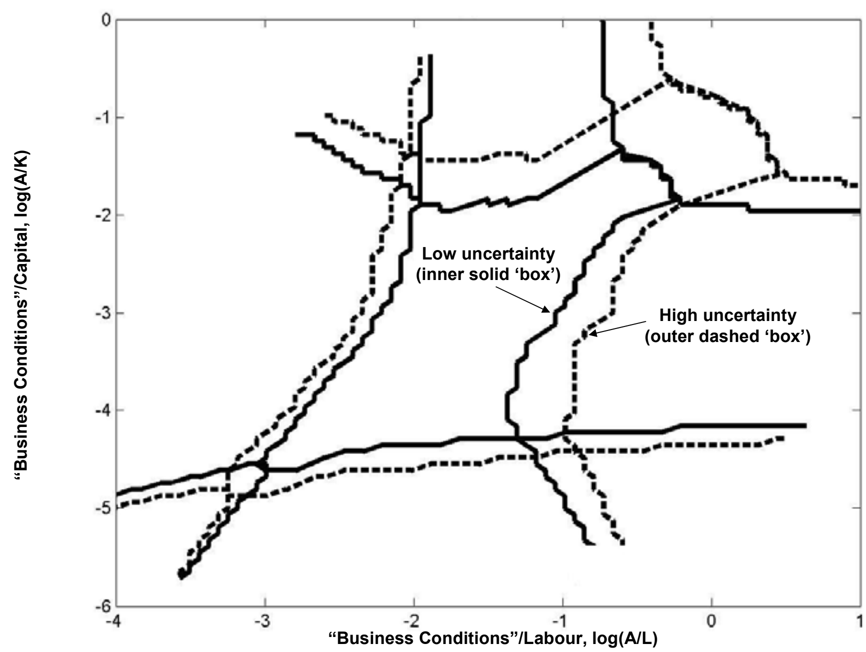
[Bloom's Figure 5 & Figure 8 here]

Figure 4: Hiring/firing and investment/disinvestment thresholds



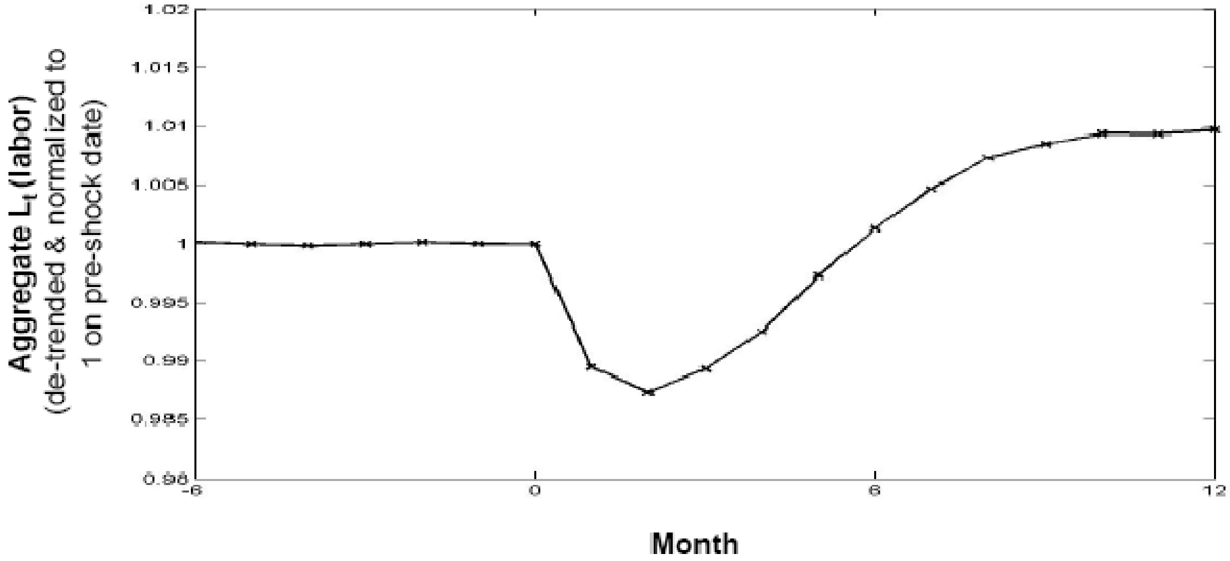
Notes: Simulated thresholds using the adjustment cost estimates “All” in table 3. All other parameters and assumptions as outlined in sections 3 and 4. Although the optimal policies are of the (s,S) type it can not be proven that this is always the case.

Figure 5: Thresholds at low and high uncertainty



Notes: Simulated thresholds using the adjustment cost estimates “All” in Table 3. All other parameters and assumptions as outlined in sections 3 and 4. High uncertainty is twice the value of low uncertainty ($\sigma_H=2\times\sigma_L$).

Figure 8: Aggregate (detrended) labor drops, rebounds and overshoots



Notes: Simulation run on 1000 units. This is repeated 25000 times with the average plotted here. All micro and macro shocks drawn randomly except at month 0, when all simulations have σ_t set to σ_H. Adjustment costs are taken from the "All" values in table 3. All other parameters and assumptions as outlined in sections 3 and 4. The aggregate figures for L_t are calculated by summing up across all units within the simulation. They are detrended by removing their long-run growth rate. The month is normalized to zero at the date of the uncertainty shock.

Source: Bloom, 2008.

5 Future research on investment

- I think it's fair to say many economists feel the regression-based approach (Euler, average Q) for analyzing investment is not satisfactory:
  - Hard to defend exogeneity of the explanatory variables;
  - To obtain a model suitable for linear regression analysis you need to make a lot of unattractive assumptions (e.g. constant returns, perfect competition etc.);
  - Empirical performance is often disappointing (e.g. q-model implies implausibly high adjustment costs; Euler equations generate results that are inconsistent with the underlying theoretical model)
- This is an area for which randomized experiments are not very suitable

- So I think Bloom's general approach will become quite popular in the literature. In fact we already see this right now.

Macroeconomics I
PhD Programme
University of Gothenburg

Reading List: Investment

Spring 2011

Måns Söderbom[1]

i) General

Chirinko, R. S. (1993). “Business Fixed Investment Spending: Modeling Strategies, Empirical Results, and Policy Implications,” Journal of Economic Literature 31, pp. 1875-1911.

Söderbom, M. (2009). “Lecture notes on Investment.” University of Gothenburg.

ii) Financial constraints

Bond, S. and M. Söderbom (2009). “Conditional Investment-Cash Flow Sensitivities and Financing constraints,” mimeo. University of Gothenburg; University of Oxford.

Fazzari, S.M., R.G. Hubbard and B.C. Petersen (1988), "Financing constraints and corporate investment", Brookings Papers on Economic Activity 1988(1):141-195.

Kaplan, S.N. and L. Zingales (1997), "Do investment-cash flow sensitivies provide useful measures of financing constraints?" Quarterly Journal of Economics 112(1):169-216.

iii) Uncertainty

Bloom, Nicholas (2009), "The Impact of Uncertainty Shocks," Econometrica 77, 623-685.

Guiso, L. and G. Parigi (1999). “Investment and Demand Uncertainty,” *Quarterly Journal of Economics*, 114(1): 185-227.

Leahy, J. and T. Whited (1996). “The Effects of Uncertainty on Investment: Some stylized facts,” *Journal of Money, Credit and Banking* 28: 64-83.

[1] University of Gothenburg. E-mail: mans.soderbom@economics.gu.se

A simple Matlab Program:

```
%{
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

This Matlab program illustrates value iteration by solving the following
optimization problem:

V(K(t)) = max PI(t) + theta*V(t+1)

where PI(t) = A^(1-beta)*[K(t) + I(t)]^beta - I(t).

We solve the problem by finding the best policy, K(t+1), given the
current state, K(t). The capital evolution formula is

K(t+1) = (1-dep)[K(t) + I(t)]

For this particular problem there exists an analytical solution for investment:

[K(t) + I(t)] = A(beta/ucc)^(1/(1-beta)) or


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%}

clear;
clc;

beta  = 0.50;            % capital elasticity in revenue function
dep   = 0.10;            % depreciation rate
r     = 0.05;            % discount rate
ucc   = (r+dep)/(1+r);
theta = 1/(1+r);
A     = 10*(beta/ucc)^(-1/(1-beta));     % Set A such that K+I=10 optimal (a nice round number)
```

```
Khatstar=A*(beta/ucc)^(1/(1-beta))*(1-dep)          % optimal K(t+1) = (1-dep)*[K(t) + I(t)]

% Next, do some housekeeping for value iteration
Knum=7;                                    % Number of points on the grid

Kstart=log(Khatstar)-1;                    % The lowest permissible value of capital
Kfinish=log(Khatstar)+1;                   % The highest permissible value of capital
Kinc=(Kfinish-Kstart)/(Knum-1);            % Implied step size

K0=exp(Kstart:Kinc:Kfinish);               % The entire vector of permissible values for capital

% Set up matrices to be used during iterations

V1=zeros(Knum,1);                      % Initial guess is a zero vector (but you could use anything)
auxV=zeros(Knum,Knum);                 % auxilary matrix to store value outcomes for different policies

% Set up the space of control variable: Capital evoluation formula Kt+1 = (1-dep)[It + Kt] implies:
% It = Kt+1/(1-dep) - Kt


I0=repmat((K0/(1-dep))',[1 Knum])-repmat(K0,[Knum 1]) ;     % Investment in t
     %  policy: K(t+1)/(1-dep)  -  state: K(t)

returns = repmat( A^(1-beta)*(K0/(1-dep))'.^beta,[1 Knum]) - I0 ;   % Cash flow in t

%returns = returns - 0.5*1*(I0./repmat(K0,[Knum 1])-dep).^2.*repmat(K0,[Knum 1])


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%     SOLVE THE MODEL BY VALUE ITERATION           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

n=1; err=1;

while err>0.0001;

   auxV = returns + theta*repmat(V1,[1 Knum]);
```

```
    [Vmax Argmax] = max(auxV);          % Vmax stores the value at the optimum choice. Argmax indexes the optimal
policy

    V2=Vmax';

    n=n+1;

    err=(V1-V2)'*(V1-V2);

    V1=V2; % Update the value function

end;
```

Table 1: The first round of the value iteration

returns: ← K(t) (state variable) →

K(t+1) (control variable in t)		3.3109	4.6208	6.4488	9	12.5605	17.5296	24.4645
	3.3109	**1.3651**	**2.6749**	**4.5029**	**7.0542**	**10.6147**	**15.5838**	**22.5187**
	4.6208	**0.224**	**1.5338**	**3.3618**	**5.9131**	**9.4736**	**14.4427**	**21.3776**
	6.4488	**-1.4359**	**-0.126**	**1.702**	**4.2532**	**7.8137**	**12.7828**	**19.7177**
	9	**-3.8319**	**-2.5221**	**-0.6941**	**1.8571**	**5.4177**	**10.3867**	**17.3217**
	12.5605	**-7.2699**	**-5.9601**	**-4.132**	**-1.5808**	**1.9797**	**6.9488**	**13.8837**
	17.5296	**-12.179**	**-10.8691**	**-9.0411**	**-6.4899**	**-2.9294**	**2.0397**	**8.9747**
	24.4645	**-19.1613**	**-17.8514**	**-16.0234**	**-13.4722**	**-9.9117**	**-4.9426**	**1.9924**

+

theta*V':

K(t+1) (state variable in t+1)								
	3.3109	**0**	**0**	**0**	**0**	**0**	**0**	**0**
	4.6208	**0**	**0**	**0**	**0**	**0**	**0**	**0**
	6.4488	**0**	**0**	**0**	**0**	**0**	**0**	**0**
	9	**0**	**0**	**0**	**0**	**0**	**0**	**0**
	12.5605	**0**	**0**	**0**	**0**	**0**	**0**	**0**
	17.5296	**0**	**0**	**0**	**0**	**0**	**0**	**0**
	24.4645	**0**	**0**	**0**	**0**	**0**	**0**	**0**

=

auxV: ← K(t) (state variable) →

K(t+1) (control variable in t)		3.3109	4.6208	6.4488	9	12.5605	17.5296	24.4645
	3.3109	**1.3651**	**2.6749**	**4.5029**	**7.0542**	**10.6147**	**15.5838**	**22.5187**
	4.6208	**0.224**	**1.5338**	**3.3618**	**5.9131**	**9.4736**	**14.4427**	**21.3776**
	6.4488	**-1.4359**	**-0.126**	**1.702**	**4.2532**	**7.8137**	**12.7828**	**19.7177**
	9	**-3.8319**	**-2.5221**	**-0.6941**	**1.8571**	**5.4177**	**10.3867**	**17.3217**
	12.5605	**-7.2699**	**-5.9601**	**-4.132**	**-1.5808**	**1.9797**	**6.9488**	**13.8837**
	17.5296	**-12.179**	**-10.8691**	**-9.0411**	**-6.4899**	**-2.9294**	**2.0397**	**8.9747**
	24.4645	**-19.1613**	**-17.8514**	**-16.0234**	**-13.4722**	**-9.9117**	**-4.9426**	**1.9924**

Key lines in the program:

```
auxV = returns + theta*repmat(V1,[1 Knum]);
[Vmax Argmax] = max(auxV)
```

Results from the max(.) command:

```
Vmax =
  1.3651   2.6749   4.5029   7.0542  10.6147  15.5838  22.5187

Argmax =
  1   1   1   1   1   1   1
```

Argmax tells me which element in the policy vector K0 is optimal. The firm has no value beyond the current time period, which is why it is optimal to sell off capital.

The crucial output for the value iteration is Vmax, however. Vmax gives me the **value of the firm** zero value beyond the current point in time, as a function of initial capital:

```
>> [K0' V2 ]

ans =

    3.3109    1.3651
    4.6208    2.6749
    6.4488    4.5029
    9.0000    7.0542
   12.5605   10.6147
   17.5296   15.5838
   24.4645   22.5187
```

Recall that my initial guess for the value function was a zero vector. Hence we have not yet converged:

```
err=(V1-V2)'*(V1-V2)
```

```
err =

  941.6717
```

Now continue to iterate on the value function, using V2 as our updated 'guess' of the true value function:

```
V1=V2; % Update the value function
```

See Table 2 for an analysis of policies, states and values using our updated guess.

Table 2: The second round of the value iteration

returns:

		← K(t) (state variable) →						
		3.3109	4.6208	6.4488	9	12.5605	17.5296	24.4645
↑ K(t+1) (control variable in t) ↓	3.3109	**1.3651**	**2.6749**	**4.5029**	**7.0542**	**10.6147**	**15.5838**	**22.5187**
	4.6208	**0.224**	**1.5338**	**3.3618**	**5.9131**	**9.4736**	**14.4427**	**21.3776**
	6.4488	**-1.4359**	**-0.126**	**1.702**	**4.2532**	**7.8137**	**12.7828**	**19.7177**
	9	**-3.8319**	**-2.5221**	**-0.6941**	**1.8571**	**5.4177**	**10.3867**	**17.3217**
	12.5605	**-7.2699**	**-5.9601**	**-4.132**	**-1.5808**	**1.9797**	**6.9488**	**13.8837**
	17.5296	**-12.179**	**-10.8691**	**-9.0411**	**-6.4899**	**-2.9294**	**2.0397**	**8.9747**
	24.4645	**-19.1613**	**-17.8514**	**-16.0234**	**-13.4722**	**-9.9117**	**-4.9426**	**1.9924**

+

theta*V':

↑ K(t+1) (state variable in t+1) ↓	3.3109	**1.3001**	**1.3001**	**1.3001**	**1.3001**	**1.3001**	**1.3001**	**1.3001**
	4.6208	**2.5475**	**2.5475**	**2.5475**	**2.5475**	**2.5475**	**2.5475**	**2.5475**
	6.4488	**4.2885**	**4.2885**	**4.2885**	**4.2885**	**4.2885**	**4.2885**	**4.2885**
	9	**6.7182**	**6.7182**	**6.7182**	**6.7182**	**6.7182**	**6.7182**	**6.7182**
	12.5605	**10.1092**	**10.1092**	**10.1092**	**10.1092**	**10.1092**	**10.1092**	**10.1092**
	17.5296	**14.8417**	**14.8417**	**14.8417**	**14.8417**	**14.8417**	**14.8417**	**14.8417**
	24.4645	**21.4464**	**21.4464**	**21.4464**	**21.4464**	**21.4464**	**21.4464**	**21.4464**

=

auxV:

		← K(t) (state variable) →						
		3.3109	4.6208	6.4488	9	12.5605	17.5296	24.4645
↑ K(t+1) (control variable in t) ↓	3.3109	**2.6651**	**3.975**	**5.803**	**8.3542**	**11.9147**	**16.8838**	**23.8187**
	4.6208	**2.7715**	**4.0813**	**5.9094**	**8.4606**	**12.0211**	**16.9902**	**23.9251**
	6.4488	**2.8526**	**4.1625**	**5.9905**	**8.5417**	**12.1022**	**17.0713**	**24.0062**
	9	**2.8863**	**4.1961**	**6.0242**	**8.5754**	**12.1359**	**17.105**	**24.0399**
	12.5605	**2.8393**	**4.1491**	**5.9772**	**8.5284**	**12.0889**	**17.058**	**23.9929**
	17.5296	**2.6627**	**3.9726**	**5.8006**	**8.3518**	**11.9123**	**16.8814**	**23.8163**
	24.4645	**2.2851**	**3.5949**	**5.423**	**7.9742**	**11.5347**	**16.5038**	**23.4387**

We update the value function again:

Vmax =

2.8863 4.1961 6.0242 8.5754 12.1359 17.1050 24.0399

V2 =

2.8863
4.1961
6.0242
8.5754
12.1359
17.1050

```
   24.0399
```

Check if there is convergence:

```
err =

   16.1990
```

and since the value function has changed a lot, we continue to iterate on it. That is, we plug in the updated value function on the right-hand side of the Bellman equation and find the maximum using the same principles as earlier. We only stop when the difference between the value function in step j-1 and that in step j is small enough. The full value iteration in this case requires more than 100 iterations. We can print out n and err as follows:

```
ans =

    2.0000  941.6717


ans =

    3.0000   16.1990

(…)

  125.0000    0.0001


ans =

  126.0000    0.0001
```

Thus, after 126 iterations, there is convergence. The value function is as follows:

```
>> [K0' V2 ]

ans =

    3.3109   33.2356
    4.6208   34.5454
    6.4488   36.3735
    9.0000   38.9247
   12.5605   42.4852
   17.5296   47.4543
   24.4645   54.3892
```

At this point we take an interest in the optimal policy. Recall that this is provided as part of Matlab’s max(.) command – in our case, all the information we need is in the vector Argmax:

```
Argmax =
```

```
    4     4     4     4     4     4     4
```

We can then find optimal policy, i.e. K(t+1), as follows:

```
>> K0(Argmax)

ans =

    9.0000    9.0000    9.0000    9.0000    9.0000    9.0000    9.0000
```

which confirms our analytical solution above.

We can easily translate this policy into optimal investment in period t, using the capital evolution formula:

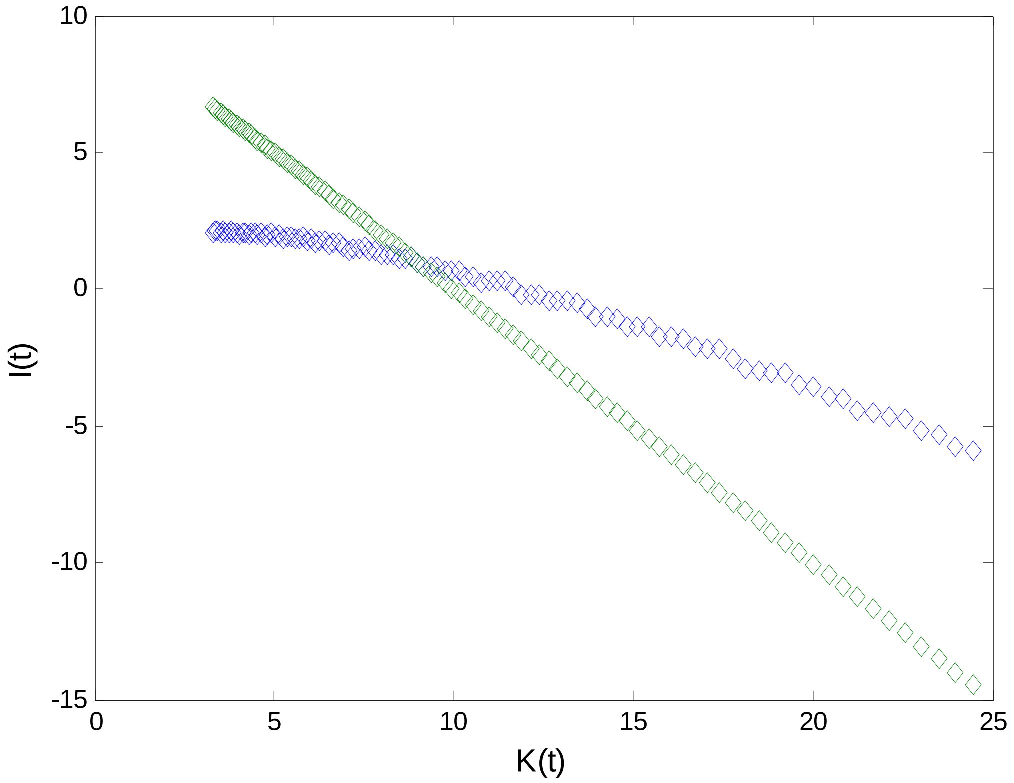
```
I0=(K0(Argmax)/(1-dep))'-K0';

>> [K0' I0]

ans =

    3.3109    6.6891
    4.6208    5.3792
    6.4488    3.5512
    9.0000    1.0000
   12.5605   -2.5605
   17.5296   -7.5296
   24.4645  -14.4645
```

The first column here is interpretable as capital in the beginning of period t; hence if you’ve got too much capital you will sell off capital and if you’ve got too little you will invest.

Generalizations:

- More points on the “grid”
- Adjustment costs
- Uncertainty

Figure 1. Investment under quadratic adjustment costs



The green line shows optimal investment under no adjustment costs. The blue line shows investment under quadratic adjustment costs, C = 0.5*0.25*[I(t)/K(t) – dep/(1-dep)]^2 * K(t).